Exam D0308
Matrix methods
Monday January 16, 2012 Time: kl. 9.00 - 13.00

The final exam consists of: 4 tasks.
Two pages excluding front page
The exam counts 100% of the final grade.
Available aids: pen and paper

Teacher: PhD David Di Ruscio
Systems and Control Engineering
Department of technology
Telemark University College
N-3914 Porsgrunn

Task 1 (25%):
The four fundamental subspaces

Assume given a matrix, $A \in \mathbb{R}^{m \times n}$, and a linear equation, $Ax = b$, where vectors, x, and, b, have compatible dimensions.

a) What is the dimensions of the vectors x and b ?

b) What is meant with the rank, r, of the matrix A ?.

c) Define each of the four fundamental subspaces.

d) Specify the dimension of each of the four fundamental subspaces.

e) Give a general requirement for the linear equation, $Ax = b$, to have a unique solution x.

Task 2 (25%): Orthogonality

Assume given a matrix, $A \in \mathbb{R}^{m \times n}$.

a) Discuss the concept of orthogonality of the four fundamental subspaces.

b) Discuss the concept of projections in connection with the linear equation, $b = Ax + e$, where, e, is the error vector.
Hint: answer should include: projection matrix P, the solution $\hat{x}$, the projection of b onto the subspace of A and the error $b - A\hat{x}$.

c) Give a short description of the QR decomposition of the matrix, A.

d) Consider a linear equation, $b = Ax + e$, where, $A \in \mathbb{R}^{m \times n}$, and $m > n$.
Show how the QR decomposition of the concatenated matrix

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_{11} & R_{21} \\ 0 & R_{22} \end{bmatrix}, \tag{1}$$

can be used to find the least squares solution, $\hat{x}$, to x?

Task 3 (25%): Singular Value Decomposition (SVD), norms and linear regression

a) Discuss the singular value decomposition of a matrix, $A \in \mathbb{R}^{m \times n}$.

b)

- Explain what is meant with the length (or norm), $\|E\|$, of a vector, $E \in \mathbb{R}^m$.
- Explain what is meant with the Frobenius norm, $\|E\|_F$, of a matrix, $E \in \mathbb{R}^{m \times n}$.

c) Consider a linear equation, $Y = XB + E$, where, $X \in \mathbb{R}^{N \times n}$, and $N > n$ and, $r = \text{rank}(X) < n$, and where we assume that Y is a vector.

Show how the Singular Value Decomposition (SVD) of the matrix, X, can be used to find the Principal Component regression (PCR) estimate, $\hat{B}_{\text{PCR}}$ of B.
Hint: The solution should minimize the squared length (or Frobenius norm), $\|E\|^2 = \|E\|_F^2$ when the error $E = Y - XB$ is a vector, and where the estimated error is $\hat{E} = Y - X\hat{B}_{\text{PCR}}$.

Task 4 (25%): Eigenvalues and the QR method

Assume given a square matrix, $A \in \mathbb{R}^{n \times n}$.

a) Discuss and define the eigenvalue decomposition of the matrix A.
Tips: answer should include eigenvalues, eigenvectors, the eigenvalue matrix, Λ, and the eigenvector matrix, S.

b)

- What is the eigenvalues of the transpose A^T of the matrix A ?
- Define the trace, trace(A), as a function of the n eigenvalues of matrix A?
Definition: The sum of the entries of the main diagonal is called the trace of A, i.e. trace(A).
- Define the determinant, det(A), as a function of the n eigenvalues of the matrix A?

c) Discuss the QR method for calculating the eigenvalues.