# Partial Test 2006

## Task 1: PID-control, the Skogestad method

***a) Find an expression for the transfer function,*** $h_c(s)$***, for the controller as a function of the ratio*** $\frac{y}{r}$ ***and the transfer function for the process,*** $h_p(s)$***.***

**ANS:**

$$\frac{y}{r} = \frac{h_p h_c}{1 + h_p h_c}$$

$$\frac{y}{r}(1 + h_p h_c) = h_p h_c$$

$$\frac{y}{r} = (1 - \frac{y}{r}) h_p h_c$$

$$h_c = \frac{\frac{y}{r}}{(1 - \frac{y}{r}) h_p}$$

$$\boxed{h_c(s) = \frac{1}{h_p} \frac{\frac{y}{r}}{(1 - \frac{y}{r})}}$$

When $\boxed{\frac{y}{r} = \frac{1 - \tau s}{1 + T_c s}}$, $\quad h_c(s) = \frac{1}{h_p} \frac{\frac{1 - \tau s}{1 + T_c s}}{(1 - \frac{1 - \tau s}{1 + T_c s})} = \frac{1}{h_p} \frac{1 - \tau s}{(T_c + \tau) s}$

*b)* ***Assume that the process,*** $h_p(s)$***, is modeled by a 2nd order transfer function given by*** $h_p(s) = k \frac{1 - \tau s}{(1 + T_1 s)(1 + T_2 s)}$ ***where*** $T_1 > T_2 > 0$***. Find the controller*** $h_c(s)$ ***by the Skogestad method.***

**ANS:**

Skogestad method: $\boxed{h_c = K_p \frac{1 + T_i s}{T_i s}(1 + T_d s)}$

$$h_c = \frac{1}{h_p}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{(1+T_1 s)(1+T_2 s)}{1-\tau s}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k(T_c+\tau)}\frac{(1+T_1 s)(1+T_2 s)}{s} = \frac{T_1}{k(T_c+\tau)}\frac{1+T_1 s}{T_1 s}(1+T_2 s)$$

$$h_c = K_p \frac{1+T_i s}{T_i s}(1+T_d s)\text{, where } K_p = \frac{T_1}{k(T_c+\tau)},\ T_i = T_{1,}\quad T_d = T_2$$

This is PID controller

***c) Assume that the process, $h_p(s)$, is modeled by a 1st order transfer function given by $h_p(s) = k\frac{1-\tau s}{1+T_1 s}$. Find the controller $h_c(s)$ by the Skogestad method.***

**ANS:**

$$h_c = \frac{1}{h_p}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{1+T_1 s}{1-\tau s}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k(T_c+\tau)}\frac{1+T_1 s}{s} = \frac{T_1}{k(T_c+\tau)}\frac{1+T_1 s}{T_1 s}$$

$$h_c = K_p \frac{1+T_i s}{T_i s}\text{, where } K_p = \frac{T_1}{k(T_c+\tau)},\ T_i = T_{1,}$$

This is PI controller

***d) Assume that the process, $h_p(s)$, is modeled by a 2nd order oscillating transfer function given by $h_p(s) = k\frac{1-\tau s}{\tau_0^2 s^2 + 2\tau_0 \xi s + 1}$. Find the controller $h_c(s)$ by the Skogestad method.***

**ANS:**

$$h_c = \frac{1}{h_p}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{\tau_0^2 s^2 + 2\tau_0\xi s + 1}{1-\tau s}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k(T_c+\tau)}\frac{\tau_0^2 s^2 + 2\tau_0\xi s + 1}{s}$$

PID controller on idea form: $\boxed{h_c = K_p(1+\frac{1}{T_i s}+T_d s)}$,

$$h_c = \frac{1}{k(T_c+\tau)}\frac{\tau_0^2 s^2 + 2\tau_0\xi s + 1}{s} = \frac{2\tau_0\xi}{k(T_c+\tau)}(1+\frac{\tau_0}{2\xi}s+\frac{1}{2\tau_0\xi s}) = K_p(1+\frac{1}{T_i s}+T_d s)$$

Where, $K_p = \frac{2\tau_0\xi}{k(T_c+\tau)}\quad T_i = \frac{1}{2\tau_0\xi}\quad T_d = \frac{\tau_0}{2\xi}$

This is PID controller

***e) Assume that process is modeled by a pure time delay, i.e. with a process model $h_p(s) = ke^{-\tau s}$ find the controller $h_c(s)$ by the Skogestad method.***

**ANS:**

Based on Talyor sequence, $e^{-\tau s} = 1 - \tau s$

$$h_c = \frac{1}{h_p}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{1}{e^{-\tau s}}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{1}{1-\tau s}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k(T_c+\tau)}\frac{1}{s}$$

This is I-controller.

***f) Assume that the process, $h_p(s)$, is modeled by a 1st order transfer function given by $h_p(s) = k\frac{1}{1+T_1 s}$. Find the controller $h_c(s)$ by the Skogestad method.***

**ANS:**

$$h_c = \frac{1}{h_p}\frac{1-\tau s}{(T_c+\tau)s} = \frac{1}{k}\frac{1+T_1 s}{1}\frac{1-\tau s}{(T_c+\tau)s} = \frac{T_1}{k(T_c+\tau)}\frac{1+T_1 s}{T_1 s}(1-\tau s)$$

$h_c = K_p \frac{1+T_i s}{T_i s}(1+T_d s)$ where, $K_p = \frac{T_1}{k(T_c+\tau)}$, $T_i = T_1$, $T_d = -\tau$ (without time delay, so $\tau = 0, \Rightarrow T_d = -\tau = 0$ )

This is PI-controller.

***g) Suggest a value for the specified time constant $T_c$ for the set point response.***

**ANS:**

$$T_c \geq \tau \quad \text{or simply} \quad T_c = \tau$$

# Task 2: Model reduction and the half rule

Given a 5[th] order process $y = h_p(s)u$ where the process transfer function, $h_p(s)$, is given by

$$h_p(s) = k\frac{1}{(1+T_1 s)(1+T_2 s)(1+T_3 s)(1+T_4 s)(1+T_4 s)} \quad \text{where } T_1 > T_2 > T_3 > T_4 > T_5 > 0$$

***a) Use the half rule for model reduction and find a 1st order model approximation of the form***

$$h_p(s) = k\frac{1-\tau s}{1+T_1 s}$$

**ANS:**

$$T_1 := T_1 + \frac{1}{2}T_2$$

$$\tau := \tau + \frac{1}{2}T_2 + T_3 + T_4 + T_5$$

**b) *Use the half rule for model reduction and find a 2nd order model approximation of the form***

$$h_p(s) = k\frac{1-\tau s}{(1+T_1 s)(1+T_2 s)}$$

**ANS:**

$$T_1 := T_1$$

$$T_2 := T_2 + \frac{1}{2}T_3$$

$$\tau := \tau + \frac{1}{2}T_3 + T_4 + T_5$$

# Task 3: System theory

a) ***Write down an expression for the solution x(t) of the state equation, for t>$t_0$***

**ANS:**

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}$$

b) ***From the result in step 3a) above find an exact discrete time model of the form*** $x_{k+1} = \Phi x_k + \Delta u_k$ ***. Specify expressions for*** $\Phi$ ***and*** $\Delta$

**ANS:**

Define $\Phi = e^{At}$, $\Delta = \int_{t_0}^{t} e^{A(t-\tau)}Bd\tau = \int_{0}^{t-t_0} e^{A\tau}Bd\tau = A^{-1}(e^{A(t-t_0)} - I)B$

$$x(t) = \Phi(t-t_0)x(t_0) + \Delta u(t_0)$$
$$x(t+\Delta t) = \Phi(\Delta t)x(t_0) + \Delta u(t_0) \qquad t_0 = k\Delta t$$
$$x((1+k)\Delta t) = \Phi(\Delta t)x(k\Delta t) + \Delta u(k\Delta t)$$
$$x_{k+1} = \Phi x_k + \Delta u_k$$

c) ***Assume now that the state equation is discretized with an explicit Euler approximation. Find a discrete time model in this case.***

**ANS:**

$$\frac{x_{k+1}-x_k}{\Delta t}=Ax_k+Bu_k$$
$$x_{k+1}=(I+A\Delta t)x_k+B\Delta t u_k$$
$$y_k=Dx_k$$

d) ***The transition matrix*** $\Phi=e^{A\Delta t}$ ***can be computed from an eigenvalue decomposition of A. Write down a formula for*** $\Phi$ ***in this case.***

**ANS:**

$\det(\lambda I-A)=0$ eigenvalues can be calculated

$A=M\Lambda M^{-1},\Phi=e^{A\Delta t}\Rightarrow\Phi=Me^{\Lambda\Delta t}M^{-1}$

Where M is the eigenvector. And $e^{\Lambda\Delta t}=\begin{bmatrix} e^{\lambda_1\Delta t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2\Delta t} & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & e^{\lambda_n\Delta t} \end{bmatrix}$

# Task 4: PI control

*Continuous state space model:*

$u=K_p\frac{1+T_is}{T_is}e=\frac{K_p}{T_is}e+K_pe=z+K_pe$ where $z=\frac{K_p}{T_is}e$

And $z=\frac{K_p}{T_is}e\Rightarrow\dot{z}=\frac{K_p}{T_i}e$

Then, $\begin{cases}\dot{z}=\frac{K_p}{T_i}e \\ u=z+K_pe\end{cases}$ and , $\begin{cases}\dot{z}=Az+Be \\ u=Dz+Ee\end{cases}$ where A=0, $B=\frac{K_p}{T_i}$, D=1, $E=K_p$

*Discrete time state space model:*

$$\begin{cases}\frac{z_{k+1}-z_k}{\Delta t}=Az_k+Be_k\Rightarrow z_{k+1}=(I+A*\Delta t)z_k+B\Delta te_k \\ u_k=Dz_k+Ee_k\end{cases}$$