Master study
Systems and Control Engineering
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SCE1106 Control Theory

Solution Exercise 5

Task 1

In connection with controller and observer canonical forms we need a matrix M formed from the coefficients in the characteristic polynomial given by

$$|sI - A| = s^2 + a_1 s + a_2 \tag{1}$$

where $a_1 = 5$ and $a_2 = 4$ when the matrix A is as given in task 1. The M matrix is then defined as follows

$$M = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \tag{2}$$

Controllability canonical form

A given state space model is transformed to controllability canonical form by using the state transformation $x = Tz$ where

$$T = C_n \tag{3}$$

and where C_n is the controllability matrix for the system $\dot{x} = Ax + Bu$.

Controller canonical form

A given state space model is transformed to controller canonical form by using the state transformation $x = Tz$ where

$$T = C_n M \tag{4}$$

Observability canonical form

A given state space model is transformed to observability canonical form by using the state transformation $x = Tz$ where

$$T = (O_n)^{-1} \tag{5}$$

where O_n os the observability matrix of the system $\dot{x} = Ax + Bu$ and $y = Dx$.

Observer canonical form

A given state space model is transformed to observer canonical form by using the state transformation $x = Tz$ where

$$T = (O_n)^{-1}(M^T)^{-1} = (M^T O_n)^{-1} \tag{6}$$

Task 2

The observed data is organized into an input data matrix/vector U and an output data matrix/vector Y as follows

$$U = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.7 \\ 0.78 \\ 0.812 \\ 0.8248 \\ 0.8299 \end{bmatrix} \tag{7}$$

Step1

Use the MATLAB plot(U) and plot(Y) functions. Se also subplot.

Step 2

From the state space model we have that

$$y_{k+1} = \phi y_k + \delta u_k = \begin{bmatrix} y_k & u_k \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix} \tag{8}$$

By using that we have $N = 8$ observations at time instants $k = 0, 1, 2, 3, 4, 5, 6, 7$ we obtain

$$\overbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}}^{Y} = \overbrace{\begin{bmatrix} y_0 & u_0 \\ y_1 & u_1 \\ y_2 & u_2 \\ y_3 & u_3 \\ y_4 & u_4 \\ y_5 & u_5 \\ y_6 & u_6 \end{bmatrix}}^{X} \overbrace{\begin{bmatrix} \phi \\ \delta \end{bmatrix}}^{B} \tag{9}$$

Hence we have the so called normal equations

$$Y = XB \tag{10}$$

which is solved for the regression parameters B as

$$B_{OLS} = (X^T X)^{-1} X^T Y \tag{11}$$

This gives

$$B_{OLS} = \begin{bmatrix} \phi \\ \delta \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix} \tag{12}$$

Hence, the parameters in the discrete time state space model is $\phi = 0.4$ and $\delta = 0.5$.

Step 3

A state space model

$$\dot{x} = ax + bu \tag{13}$$

have the discrete time description

$$x_{k+1} = \phi x_k + \delta u_k \tag{14}$$

where

$$\phi = e^{a\Delta t} = e^{-\frac{\Delta t}{T}} \tag{15}$$

where $\Delta t = 10$ is the sampling time and $T = -\frac{1}{a}$ is the time constant in the continuous time process. Hence, we have

$$T = -\frac{\Delta t}{\ln(a)} \approx 10.91 \tag{16}$$

The steady state gain can be found by using that $x_{k+1} = x_k = x$ in steady state. This gives

$$x_k = \frac{\delta}{1-\phi} u_k = 0.8333 u_k \tag{17}$$

Hence, the steady state gain is given by $y_k = h^d u_k$ where the steady state gain is

$$h^d = 0.8333 \tag{18}$$

Note also that the realationship between δ and b is

$$\delta = a^{-1}(e^{a\Delta t} - 1)b \tag{19}$$

if zero order hold is assumed, i.e., if we assume that $u(t)$ is constant over the sampling interval.

Step 4

From the above and that $\phi = e^{a\Delta t}$ we have that

$$a = \frac{\ln(\phi)}{\Delta t} = -\frac{1}{T} = -0.0916 \tag{20}$$

We solve (19) for b and get

$$b = 0.0764 \tag{21}$$

Step 5

The transfer function model from u to y is given by

$$y = h_p(s)u \tag{22}$$

where

$$h_p(s) = \frac{k}{1+Ts} \tag{23}$$

where $T = 10.9133$ is the time constant and $k = 0.8333$ is the steady state gain.

Step 6

A PI controller is represented as

$$u(s) = h_c(s)e(s) \tag{24}$$

where $e(s) = r - y(s)$ is the control deviation (controller input) and h_c the PI controller transfer function

$$h_c(s) = K_p \frac{1+T_i s}{T_i s} \tag{25}$$

in the Laplace plane.

A state space formulation of the PI controller may be as follows

$$u = z + K_p e \tag{26}$$

$$\dot{z} = \frac{K_p}{T_i} e \tag{27}$$

Step 7 We may chose $T_i = T = 10.9133$ in order to simplify the loop transfer function

$$h_o(s) = h_p(s)h_c(s) = \frac{k}{1+Ts} K_p \frac{1+T_i s}{T_i s} = \frac{kK_p}{Ts} \tag{28}$$

The transfer function from r to y is then given by

$$\frac{y}{r}(s) = \frac{h_p h_c}{1+h_p h_c} = \frac{1}{1+T_c s} \tag{29}$$

where

$$T_c = \frac{T}{kK_p} \tag{30}$$

is the time constant for the closed loop system. Choosing $T_c = 4$ gives

$$K_p = \frac{T}{kT_c} \approx 3.2740 \tag{31}$$

Note, the Skogestad method could also have been used here by specifying the set point response as

$$\frac{y}{r}(s) == \frac{h_p h_c}{1 + h_p h_c} = \frac{1}{1 + T_c s}, \tag{32}$$

and choosing $T_c = 4$ and solving for the controller, $h_c(s)$. Do this as an exercise!

Step 8

If a time delay equal to $\tau = \frac{\Delta t}{2} = 5$ is included in the observed model we obtain the following process model

$$\frac{y(s)}{u(s)} = h_p(s) = \frac{k}{1 + Ts} e^{-\tau s} \tag{33}$$

where

$$k = 0.8333, \;\; T = 10.9, \;\; \tau = 5. \tag{34}$$

Since the transport delay, $e^{-\tau s}$, is not a rational function we use the series apprximation

$$e^{-\tau s} \approx 1 - \tau s \tag{35}$$

in order to do the computations. Other approximations as illustrated in the Lecture notes, Ch. 5, could however be used instead. However, the simple approximation (35) gives simple and reasonable results. This gives the following model for PI controller synthesis

$$\frac{y(s)}{u(s)} = h_p(s) = k \frac{1 - \tau s}{1 + Ts} \tag{36}$$

Let us now specify the set point response from the reference $r = y^s$ to the output y as

$$\frac{y}{r} = \frac{h_p h_c}{1 + h_p h_c} = \frac{1 - \tau s}{1 + T_c s}, \tag{37}$$

where T_c is a specified time constant for the set point response (the closed loop system). As is suggested in the Skogestad method we usually can chose $T_c = \tau = 5$.

We now solve Equation (37) with respect to the controller transfer function, $h_c(s)$. This gives

$$h_c = \frac{1}{h_p} \frac{\frac{y}{r}}{1 - \frac{y}{r}} = \frac{1}{k} \frac{1 + Ts}{(T_c + \tau)s} = \frac{T}{k(T_c + \tau)} \frac{1 + Ts}{Ts}. \tag{38}$$

This is a PI controller of the form

$$h_c = K_p \frac{1 + T_i s}{T_i s}. \tag{39}$$

in which

$$\begin{aligned} K_p &= \frac{T}{k(T_c + \tau)} = \frac{T}{2k\tau} = 1.3081, & (40) \\ T_i &= T = 10.9. & (41) \end{aligned}$$