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On Tuning PI Controllers for Integrating Plus Time Delay Systems

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Abstract

Some analytical results concerning PI controller tuning based on integrator plus time delay models are worked out and presented. A method for obtaining PI controller parameters, $K_p = \frac{\alpha}{k\tau}$, and, $T_i = \beta\tau$, which ensures a given prescribed maximum time delay error, $d\tau_{\max}$, to time delay, τ, ratio parameter $\delta = \frac{d\tau_{\max}}{\tau}$, is presented. The corner stone in this method, is a method product parameter, $\bar{c} = \alpha\beta$. Analytical relations between the PI controller parameters, T_i, and, K_p, and the time delay error parameter, δ, is presented, and we propose the setting, $\beta = \frac{\bar{c}}{a}(\delta + 1)$, and, $\alpha = \frac{a}{\delta+1}$, which gives, $T_i = \frac{\bar{c}}{a}(\delta + 1)\tau$, and, $K_p = \frac{a}{(\delta+1)k\tau}$, where the parameter, a, is constant in the method product parameter, $\bar{c} = \alpha\beta$. It also turns out that the integral time, T_i, is linear in, δ, and the proportional gain, K_p, inversely proportional to, $\delta + 1$. For the original Ziegler Nichols (ZN) method this parameter is approximately, $\bar{c} = 2.38$, and the presented method may e.g., be used to obtain new modified ZN parameters with increased robustness margins, also documented in the paper.

Keywords: PI controller, tuning, integrating system, time delay, maximum time delay error, frequency analysis

1. Introduction

This paper is concerned about PI controller tuning based on integrator plus time delay models. Integrator plus time delay processes and close to integrator plus time delay systems are common and important processes in industry. Examples of integrating plus time delay processes are level systems, pulp and paper plants, oil-water-gas separators in oil industry, and all time constant lag dominant processes which may be approximated with an integrator plus time delay process. Reported examples are high purity distillation columns where there are large time constants for small changes in the set-point, and where the delay e.g. comes from an analyzer, see e.g. Chien and Fruehauf (1990), Tyreus and Luyben (1992) and earlier references in this paper. In Skogestad (2001), Sec. 6.4 of that paper, it is also stated that integrating processes are important in industry and an example of re-boiler control in connection with a distillation column is presented.

Most PI controller tuning rules for integrating plus time delay processes may be formulated in the following setting

$$K_p = \frac{\alpha}{k\tau}, \; T_i = \beta\tau, \tag{1}$$

where, K_p, is the PI controller proportional gain, T_i, the integral time, k, is the gain velocity or the slope of the integrator and, τ, is the time delay. Here, α, and, β, are dimensionless parameters, which may be related to each other, e.g. such that β is a function of α or vice versa. For instance using the classical tuning rules by Ziegler and Nichols (1942), $\alpha = 0.71$ and $\beta = 3.33$. Using the IMC tuning rules in Table 1 of Chien and Fruehauf (1990) with closed loop time constant, $\tau_{cl} = \sqrt{10}\tau$, as proposed in Tyreus and Luyben (1992) gives parameters $\alpha = 0.42$ and $\beta = 7.32$. Using the Simple Internal Model Control (SIMC) Skoges-

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tad (2001) tuning rules with closed loop time constant, $T_c = \tau$, gives $\alpha = 0.5$ and $\beta = 8$. This also holds for the tuning rules deduced in Chidambaram and Sree (2003).

In order to obtain PI controller settings with good robustness properties and at the same time reasonable fast set-point and disturbance properties, for integrating plus time delay processes, without e.g. too much overshoot, then the size and balanced relationship between the two parameters α and β are of appropriate importance.

From the basic PI setting in eq. (1) we may also define a method product parameter, $\bar{c}$, for later use, as

$$\bar{c} = \alpha\beta. \tag{2}$$

Notice also that $\bar{c} = \alpha\beta = K_p k T_i$ holds for the setting in eq. (1). The defined method product parameter, $\bar{c}$, in eq. (2) is constant for many methods. The SIMC PI settings, Skogestad (2001), yield a method product parameter, $\bar{c} = \alpha\beta = 4$. The original Ziegler Nichols (ZN) method gives a parameter approximately, $\bar{c} = \alpha\beta = 2.38$.

This work is somewhat inspired by the Skogestad (2001) SIMC PID controller tuning rules, also further presented in Skogestad (2003) and Skogestad (2004). The SIMC PI controller settings for a pure integration plus time delay process is derived in order to obtain simple and robust tuning rules, i.e. by using the proportional gain setting found from an argumentation that the process is first order plus time delay, and time constant/lag dominant. Furthermore, studying the set-point response we find a P-controller equivalent to the SIMC setting. In order to suppress input disturbances a PI controller is needed for integrating plus time delay processes. The integral time constant in the SIMC PI setting for pure integration plus time delay processes is found by neglecting the time delay and specifying a pole polynomial for the responses with unit relative damping. The resulting margins are relatively good, but somewhat reduced compared to the good margins when using the SIMC PI controller settings for a first order plus time delay process, or using a pure P controller for integrating plus time delay processes (Gain Margin equal to π and Phase Margin equal to $61°$).

In the IMC settings, Chien and Fruehauf (1990), and in the SIMC settings, Skogestad (2001, 2003, 2004), the closed loop time constant is the tuning parameter. We believe that there in general will be some trial and error procedure involved in prescribing the closed loop time constant. When using the SIMC method the closed loop time constant, T_c, is the tuning parameter in the range $-\tau < T_c < \infty$, and for robust tuning $T_c \geq \tau$, Skogestad (2001). However, for fast and robust control it is recommended to choose the simple choice indeed, i.e., $T_c = \tau$, Skogestad (2003). See also Shamsuzzoha et al. (2010) for a statement of this. We notice in connection with this, that a basic requirement when choosing the tuning parameter should be stability of the closed loop system. In Tyreus and Luyben (1992) it was commented upon that the IMC approach requires some trial and error in order to specify the closed loop time constant that will give a reasonable damping in the closed loop responses.

The main foci and motivations of this paper may be itemized as follows:

- One of the main foci in this paper is to discuss PI controller tuning rules for integrating plus time delay systems, and instead choose the closed loop time constant such that some sensitivity or robustness measure is achieved. In connections with such systems it makes sense to focus on the phase margin and the corresponding maximum time delay error, which also is one of the main foci of this paper.

- The disturbance response by using the SIMC PI controller settings is relatively slow and may possibly be improved, without reducing the margins. This problem is among others addressed in this paper.

- A question also investigated in this paper is whether we may deduce PI controller tuning rules for integrating plus time delay systems in which the method product parameter, is less than $\bar{c} = 4$, without reducing margins and with approximately the same set-point and disturbance response properties, as e.g. the SIMC method with the robust lower bound for fast response, i.e. $T_c = \tau$.

Notice in connection with this, that one also may focus on the maximum peak M_s, of the sensitivity function as also described in Åström and Hägglund (2004), and Åström and Hägglund (1995) where some inequalities relating the gain margin and phase margin, PM, to the M_s sensitivity index are presented on p. 126. Reasonable values of the minimum sensitivity index M_s are in the range from $1.3 \leq M_s \leq 2$, Åström and Hägglund (1995). The M_s sensitivity measure is noticed upon in connection with some of the results and examples in this paper, and may be a topic for further research.

The contributions of this paper may be itemized as follows:

- A method for tuning PI controller parameters, α, and, β, in eq. (1) based on integrator plus time delay models, such that the resulting closed loop system obtains a prescribed maximum time delay error, $d\tau_{\max}$, to time delay, τ, ratio $\delta = \frac{d\tau_{\max}}{\tau}$, is presented in Sec. 6. This method has two tuning parameters, in addition to the maximum time delay error ratio, δ, a method tuning parameter, $\bar{c}$.

- Instead of neglecting the time delay when deriving the integral time constant, $T_i = \min(T, 4(T_c + \tau)) = 4(T_c + \tau)$, as in the SIMC method we, in Sec. 5, use the truncated series approximation, and common approximations as Pade and Balchen approximations, and derive alternative relations between the proportional gain, K_p, and T_i, and alternative PI controller settings are derived. This setting gives somewhat improved margins compared with the SIMC PI controller tuning, faster disturbance response and approximately the same set-point response.

- We discuss alternative settings for the integral time constant, e.g., $T_i = \min(T, 3(T_c + \tau))$, in the SIMC method for PI control of integrator plus time delay processes which gives a faster input disturbance response than the SIMC setting, but with approximately the same margins.

- We propose a PI controller setting, for first order plus time delay systems, where instead of specifying the time constant of the set-point response as in the SIMC method, use a dimensionless parameter, c, such that the closed loop time constant is, $T_c = c\tau$, and such that the closed loop system gets a prescribed Gain Margin, GM.

- A review over some existing PI controller tuning rules for integrating plus time delay processes are given.

The rest of this paper is organized as follows. In Sec. 2 some basic theory and definitions used throughout the paper are presented. In Sec. 3 we give a discussion of how to find the PI controller parameters such that the closed loop system gets a prescribed Gain margin, for systems in which the integral time, T_i, is chosen equal to the dominant time constant. In Sec. 4 we work through the SIMC method, Skogestad (2001), for integrator plus time delay systems, as well as the SIMC settings $T_i = \min(T, 4(T_c + \tau)) = 4(T_c + \tau)$ with some connected discussions. In Sec. 5 we derive PI controller tuning rules, but instead of neglecting the time delay as when deriving the SIMC rules in Sec. 4, we instead are using three different approximations to the time delay, i.e., an inverse response approximation, a Pade approximation and a time lag approximation. In Sec. 6 we deduce analytical results concerning the maximum time delay error and propose PI controller tuning rules in terms of a prescribed maximum time delay error. Simulation examples are presented in Sec. 7. Some related discussions are given in Sec. 8 and conclusions follow in Sec. 9.

2. Preliminary Theory

2.1. Lag Dominant Systems

Consider a system approximated with a first order time constant plus time delay model

$$h_p(s) = K\frac{e^{-\tau s}}{1 + Ts}, \tag{3}$$

where, K, is the process gain, T the dominant time constant or time lag and, τ, the time delay.

The system eq. (3) may be defined as lag dominant when $T > \tau$ which is the case for many systems. It is well known that when $T \gg \tau$ then eq. (3) may be approximated with a pure integrator plus time delay model and the controller tuning could be based on this approximation, Chien and Fruehauf (1990), Tyreus and Luyben (1992).

From the model eq. (3) we obtain

$$h_p(s) = \frac{K}{T}\frac{e^{-\tau s}}{s + \frac{1}{T}}, \tag{4}$$

where we define the factor between the gain, K, and the time constant/lag, T, as the gain velocity, $k = \frac{K}{T}$ When the system is lag dominant and T "large" we may approximate eq. (4) as an integrator plus time delay system

$$h_p(s) = k\frac{e^{-\tau s}}{s}, \tag{5}$$

where $k = \frac{K}{T}$ is the slope of the integrator, i.e., the gain velocity and τ the time delay.

We will in this paper focus on PI controller tuning which may be based on the integrator plus time delay system eq. (5), and that this integrator plus time delay model may be an approximation of a lag dominant first order plus time delay model as in eq. (3).

Furthermore, notice that if the SIMC method for PI controller tuning as in Skogestad (2001) is used, then the PI controller tuning becomes the same, whether the tuning is based on the lag dominant model eq. (3) or the integrator plus time delay model approximation eq. (5), when $\min(T, 4(T_c + \tau)) = 8\tau$. We have here assumed the lower bound for the closed loop time constant, $T_c = \tau$. This means that we may tune the PI controller based on eq. (5) with gain velocity $k = \frac{K}{T}$ when $T > 8\tau$, when e.g. the SIMC method is used.

This also implies that most methods which are constructed for integrating plus time delay systems may work well for time lag dominant systems. The main focus of this paper is to possibly give some improvements of PI controller tuning for such systems.

2.2. SIMC Tuning Rules for First Order Plus Time Delay Process

Consider the first order time constant plus time delay process in eq. (3). The standard SIMC PI controller settings (Skogestad (2001), Skogestad (2003)) for the PI controller parameters are

$$K_p = \frac{T}{K(T_c + \tau)}, \ \ T_i = \min(T, 4(T_c + \tau)), \tag{6}$$

where $T_c \geq \tau$ for robust tunings Skogestad (2001), is a prescribed time constant for the set-point response.

Consider the case in which $\min(T, 4(T_c + \tau)) = T$. Canceling the dominant time constant by choosing, $T_i = T$, gives the PI controller transfer function, $h_c(s) = K_p \frac{1+T_i s}{T_i s}$, with proportional gain as in eq. (6). This is found by specifying the loop-transfer function, $\frac{y}{r}(s) = \frac{h_c h_p}{1+h_c h_p} = \frac{e^{-\tau s}}{1+T_c s}$, and solving for the controller, h_c, which gives, $h_c(s) = \frac{1}{h_p(s)} \frac{\frac{y}{r}(s)}{1-\frac{y}{r}(s)}$, and in order to ensure a rational controller transfer function, $h_c(s)$, the approximation, $e^{-\tau s} = 1 - \tau s$, is used. The case when the minimum time constant is, $T_i = 4(T_c + \tau)$, is derived based on an integrating plus time delay process as discussed in Subsection 4.2.

2.2.1. Margins for the SIMC PI Setting: First Order Plus Time Delay

We will in this section discuss the guaranteed margins for the SIMC PI controller settings for a first order plus time delay process. The aim is to present some definitions used in the paper.

Consider the robust lower bound and simple choice $T_c = \tau$ which gives $K_p = \frac{T}{2K\tau}$, and the case with $T_i = \min(T, 8\tau) = T$. As in Appendix A we find the gain margin $GM = \frac{1}{|h_0(j\omega_{180})|} = \pi$. Assume that the true process gain is, k_p, and different from our model gain K. Then this means that we may tolerate a multiplicative uncertainty in the process gain, $k_p = GMK$, (at the phase crossover frequency, ω_{180}) before the system becomes unstable.

The SIMC PI controller tunings ($T \leq 8\tau$) give a constant gain margin $GM = \pi$ irrespective of the model parameters K, T and τ. For the setting $T_c = \tau$ and $T_i = \min(T, 8\tau) = 8\tau$ then the gain margin is approximately equal to 3, also reported in Shamsuzzoha et al. (2010). This case is discussed in detail based on an integrator plus time delay model, in Sec. 4 and Sec. 6, where some results regarding the Phase margin and maximum time delay error are derived.

The SIMC PI controller tuning yields a Phase Margin, $PM \approx 61.4°$, as described in Appendix A. Furthermore we may tolerate a maximum time delay error,

$$d\tau_{\max} = \frac{PM}{\omega_c} = (\pi - 1)\tau = 2.14\tau. \tag{7}$$

One interpretation of this is as follows. Suppose that the true time delay, τ_p, in the process is, $\tau_p = \tau + d\tau$, where τ is the time delay in the model. The corresponding true Phase Margin is then $PM_p = -(\tau + d\tau)\omega_c - \frac{\pi}{2} + \pi = PM - d\tau\omega_c$. The maximum time delay error perturbation, $d\tau_{\max}$, which may be tolerated before the system becomes unstable is found for the phase margin limit ($PM_p = 0$), i.e., $PM_p = PM - d\tau_{\max}\omega_c = 0$, which gives eq. (7).

2.3. On Some Methods for Tuning Integrating Plus Time Delay Systems

2.3.1. Tyreus and Lyben Modified ZN Tuning

The Tyreus and Lyben (TL) settings as presented in Tyreus and Luyben (1992) are re-presented for comparison purposes in Skogestad (2001)-Skogestad (2004), but the parameters settings may be misunderstood. In Tyreus and Luyben (1992) it is suggested

$$K_p = \frac{K_u}{3.22} = 0.311K_u, \ \ T_i = 2.2P_u, \tag{8}$$

where K_u is the ultimate gain and P_u the ultimate period. For an integrator plus time delay process with P-controller we simply find the ultimate period and ultimate gain from the frequency response of the corresponding loop transfer function, i.e., $h_0(j\omega) = |h_0(j\omega)|e^{\angle h_0(j\omega)}$ where the magnitude is, $|h_0(j\omega)| = \frac{K_p k}{\omega}$ and the phase angle $\angle h_0(j\omega) = -\tau\omega - \frac{\pi}{2}$. This gives the Phase crossover frequency, $\omega_{180} = \frac{\pi}{2\tau}$, such that $\angle h_0(j\omega_{180}) = -\pi$. This gives the ultimate period, $P_u = \frac{2\pi}{\omega_{180}} = 4\tau$. The ultimate gain is the largest K_p such that the magnitude $|h_0(j\omega_{180})| = 1$ which gives the ultimate gain, $K_u = \frac{\pi}{2k\tau}$, also such that the phase crossover frequency and the gain crossover frequency coincide, i.e., $\omega_c = \omega_{180}$. This gives the PI controller settings

$$K_p = \frac{0.4878}{k\tau}, \ \ T_i = 8.8\tau. \tag{9}$$

This setting gives approximately the same responses as the SIMC setting (for an integrator plus time delay process), and with a somewhat slow disturbance response, and with much improved margins compared to the ZN settings discussed below, i.e., a $GM \approx 3.06$, a Phase margin, $PM \approx 48.54°$, a maximum time delay error, $d\tau_{\max} \approx 1.69\tau$ and a sensitivity index $M_s \approx 1.67$. This is further analyzed in Example 7.1.

2.3.2. Original Ziegler Nichols (ZN) Tuning

From the above discussion in Sec. 2.3.1 we find the ZN PI controller tunings for a pure integrating plus time delay process as follows

$$K_p = \frac{K_u}{2.2} \approx \frac{0.714}{k\tau}, \; T_i = P_u/1.2 \approx 3.33\tau. \quad (10)$$

This setting gives relatively aggressive responses with oscillations and relatively poor robustness margins, i,e, a Gain margin, $GM \approx 1.85$, a Phase margin, $PM \approx 24.7°$, a maximum time delay error, $d\tau_{\max} \approx 0.56\tau$ and a sensitivity index $M_s \approx 2.86$. Notice that this ZN tuning is further used in Example 6.1 where we increase the margins by prescribing a maximum time delay error.

2.3.3. Discussion

Notice that two PI controller settings are proposed in Tyreus and Luyben (1992), as follows:

- The "TL ultimate gain and period method", Tyreus and Luyben (1992) as in eq. (9) above.
- The proposed settings in Tyreus and Luyben (1992) with closed loop time constant, $\tau_0 = \sqrt{10}\tau$, eq. (18) in that paper and in corresponding with the settings eqs. (2-3) of that paper (equivalent with IMC settings in Table 1 in Chien and Fruehauf (1990)), give the settings $K_p = \frac{0.42}{k\tau}$ and $T_i = 7.32\tau$.

Notice also that there possibly is a minor error in Table 3 in Skogestad (2001) were the TL settings are presented as $K_p = 0.49$ and $T_i = 7.32$ (for $k = \tau = 1$).

3. Gain Margin Aspects

We will later on in Sec. 6 deduce some analytical results concerning the maximum time delay error of integrator plus time delay systems, and it makes sense to focus on the Phase margin for such systems, and instead prescribe the maximum time delay error, to time delay ratio, $\frac{d\tau_{\max}}{\tau}$. However, in connections with time constant systems, as e.g. in eq. (3), where the integral time is chosen as the dominant time constant, i.e., $T_i = T$, we may look at the Gain margin and some results are discussed in the following.

3.1. Specifying Gain Margin Instead of Closed Loop Time Constant

The time constant, T_c, for the set-point and disturbance load responses may in some circumstances be difficult to specify in advance. Often some trial and error procedure is used, also commented upon in Tyreus and Luyben (1992).

Let us instead chose, T_c, as a dimensionless parameter, c, times the time delay, i.e.,

$$T_c = c\tau, \quad (11)$$

where the parameter, c, is chosen such that the feedback system has a prescribed Gain margin, GM_{pre}. We find the settings

$$K_p = \frac{T}{(c+1)K\tau}, \; T_i = \min(T, 4(c+1)\tau), \quad (12)$$

and when $T_i = T$ we simply have the parameter c as

$$c = \frac{2}{\pi} GM_{\text{pre}} - 1. \quad (13)$$

Note here, that when prescribing a Gain margin, $GM_{\text{pre}} = \pi$, then we obtain the SIMC tuning rules for a first order time delay process, with $T_i = T$, $T_c = \tau$, i.e., $K_p = \frac{T}{2K\tau}$. Typical values for the gain margin are in the range $2 \leq GM \leq 5$, Åström and Hägglund (1995) p. 126.

Notice also that the closed loop time constant now is found by using eq. (13), i.e.,

$$T_c = \left(\frac{2}{\pi} GM_{\text{pre}} - 1\right)\tau, \quad (14)$$

with $GM_{\text{pre}} > 1$ in order to ensure stability of the closed loop system. This strategy may help to find a reasonable closed loop time constant, T_c, as given in eq. (14). Notice that eqs. (12) and (13) may be combined to give K_p as a function of GM_{pre}, i.e.,

$$K_p = \frac{\pi}{2} \frac{T}{GM_{\text{pre}} K\tau}. \quad (15)$$

Note that we here have proposed an alternative approach for tuning the PI controller parameters. In the SIMC method the time constant, ($T_c \geq \tau$ for robust tunings), is specified initially, but in this gain margin approach the PI controller parameters are a function of the prescribed Gain margin, GM_{pre}. One should also notice the simple setting, eq. (11), of the closed loop set-point response time constant.

The case in which $T_i = \min(T, 4(c+1)\tau) = 4(c+1)\tau$ is not so simple as the explicit setting for, c, given by eq. (13). This case is considered in connection with integrator plus time delay systems, and we will instead focus on the Phase margin and the maximum time delay error for such systems, as discussed in Sec. 6.

Also notice that this gain margin approach gives almost similar tunings as the SIMC tuning rules, for a first order plus time delay model, the only difference is that instead of specifying the time constant, T_c, of the set-point response, the dimensionless parameter, c, which corresponds to a prescribed Gain margin, GM_{pre}, is used.

We will in the next Sections 4, 5 and 6 focus on PI controller settings for integrating plus time delay systems, in which the resulting PI controller parameter settings give some improved controller performance, both with respect to load disturbance rejections and robustness margins.

3.2. Tuning for Prescribed Gain Margin

We will in this section look at a simple PI controller tuning method which results in a prescribed gain margin, GM_{pre}. Consider the SIMC PI controller setting for the integral time, (Skogestad (2001), Skogestad (2003)), and the case in which

$$T_i = \min(T, 4\zeta^2(T_c+\tau)) = T, \tag{16}$$

where Skogestad (2001) is choosing a relative damping factor $\zeta = 1$. The case in which $T_i = \min(T, 4\zeta^2(T_c+\tau)) = 4\zeta^2(T_c+\tau)$ is focused on in Sec. 4.

Assume that we want a prescribed Gain Margin, GM_{pre}, for the feedback system. In order to find the proportional gain, K_p, which gives this gain margin we first find the gain margin, $GM_{(K_p=1)}$, for the loop transfer function with unit proportional gain, i.e. a PI controller with $K_p = 1$,

$$h_0(s) = \frac{1+T_i s}{T_i s} h_p(s). \tag{17}$$

The proportional gain

$$K_p = \frac{GM_{(K_p=1)}}{GM_{\text{pre}}}, \tag{18}$$

ensures that the loop transfer function obtains the prescribed gain margin, GM_{pre}. We have in the above assumed that the system is stable for a unit proportional gain in connection with this result, and that the open loop system is stable with real time constants. This is not considered further.

4. On the SIMC Method for Integrating Plus Time Delay Process

4.1. Input Load Disturbance

Consider a system described by the transfer function

$$h_p(s) = k\frac{e^{-\tau s}}{s}, \tag{19}$$

which is an integrator process with time delay. Note that we may approximate eq. (19) as an inverse response with $\tau \geq 0$. Using the method as presented in Sec. 2.2 for a process, $y = h_p(s)u$, leads to a P-controller with proportional gain

$$K_p = \frac{1}{k(T_c+\tau)} = \frac{1}{2k\tau}, \tag{20}$$

where the last equality is obtained by the simple rule of thumb, $T_c = \tau$. This P-controller setting has good margins, i.e., a Gain Margin, $GM = \pi$, Phase Margin, $PM = 61^\circ$ and a maximum time delay error $d\tau_{\text{max}} = 2.14\tau$.

Unfortunately, a P-controller will give set-point error for disturbances at the input, i.e. for systems $y = h_p(s)(u+v)$ because the response from the disturbance to the output then is given by

$$y = \overbrace{\frac{h_c h_p}{1+h_c h_p}}^{\frac{y}{r}(s)} r + \overbrace{\frac{h_p}{1+h_c h_p}}^{\frac{y}{v}(s)} v. \tag{21}$$

Looking at the response from the disturbance, v, to the output, y, for a process $h_p = k\frac{e^{-\tau s}}{s}$ and a P-controller, i.e., $h_c = K_p$ gives,

$$\frac{y}{v}(s) = \frac{k\frac{e^{-\tau s}}{s}}{1+K_p k\frac{e^{-\tau s}}{s}}. \tag{22}$$

In steady-state we have $\frac{y}{v}(s=0) = \frac{1}{K_p}$ and that

$$y = r + \frac{1}{K_p} v. \tag{23}$$

This implies that we usually need a PI-controller for integrating plus time delay systems in order to eliminate the offset from load disturbances, v, at the input, i.e., we need a controller in which, $\frac{y}{v}(s=0) = 0$. Note that load disturbances at the output will be removed by using a P-controller, i.e., for systems, $y = h_p(s)u + v$, and integrating plus time delay systems as in eq. (19).

4.2. Neglecting the Time Delay when Deriving the Integral Time

In practice, for the reason of eliminating load disturbances v at the input, i.e., for systems $y = h_p(s)(u+v)$ and in case of unmodeled effects we use a PI controller for integrating processes. The SIMC PI settings, Skogestad (2001), are

$$K_p = \frac{1}{k(T_c+\tau)}, \quad T_i = 4(T_c+\tau), \tag{24}$$

and with the robust simple choice for the time constant for the set-point response, $T_c = \tau$, we obtain the SIMC settings

$$K_p = \frac{1}{2k\tau}, \; T_i = 8\tau. \tag{25}$$

This SIMC setting gives reasonable margins, i.e., a gain margin, $GM \approx 2.96$ and a Phase margin, $PM = 46.8°$.

The SIMC integral time setting in eq. (24) may be deduced as follows. Consider a first order system with time delay, and with a large time constant, T, i.e. we may write the model as

$$h_p(s) = K\frac{e^{-\tau s}}{1+Ts} = \frac{K}{T}\frac{e^{-\tau s}}{\frac{1}{T}+s} \approx k\frac{e^{-\tau s}}{s}, \tag{26}$$

with, $k = \frac{K}{T}$, the slope of the integrator step response. In the time domain, k, is the input gain parameter/matrix for a model, $\dot{y}^- = ku$, and a delay, $y = y^-(t-\tau)$. The slope/gain, k, may be found from system identification but the time delay may be more problematic in case of high frequency noise on the data, but this is not a topic of this paper. For systems with large time constants and neglecting the time delay we obtain the transfer function

$$h_p(s) \approx k\frac{1}{s}, \tag{27}$$

which is used for the derivation of the SIMC PI-settings.

In Skogestad (2001) it is argued that the oscillations caused by the delay occur at a frequency, $\omega \approx \frac{1}{\tau}$, and is faster then the "slow" oscillations caused by the disturbances, and the delay is therefore neglected in the SIMC derivation. From which is found below, the disturbance oscillations caused by a high proportional gain, K_p, occur at a frequency $\omega \approx \frac{1}{4\tau}$. Notice that this argumentation is obtained from separating the delay from the problem, and from simulation experiments.

The pole polynomial for the disturbance and set-point response is obtained from

$$\begin{aligned} 1 + h_c h_p &= 1 + K_p\frac{1+T_i s}{T_i s}\frac{k}{s} \\ &= \frac{1}{s^2}\left(s^2 + \frac{K_p k}{T_i}(1+T_i s)\right) \\ &= \frac{1}{s^2}\frac{K_p k}{T_i}\left(\frac{T_i}{K_p k}s^2 + T_i s + 1\right). \end{aligned} \tag{28}$$

This gives a pole polynomial on standard second order form as

$$\pi(s) = \frac{T_i}{K_p k}s^2 + T_i s + 1 = \tau_0^2 s^2 + 2\zeta\tau_0 s + 1, \tag{29}$$

where, τ_0, is the speed of response for a given dimensionless relative damping coefficient, ζ. Note that eq. (29) may be written in terms of the natural/resonance frequency, $\omega = \frac{1}{\tau_0}$.

By comparing the coefficients in the pole polynomial and the corresponding coefficients in the standard second order polynomial we may find relations between, K_p, and, T_i. We have

$$\tau_0^2 = \frac{T_i}{K_p k}, \; 2\zeta\tau_0 = T_i. \tag{30}$$

This gives $(2\zeta\tau_0)^2 = T_i^2$ and

$$T_i = 4\zeta^2\frac{1}{K_p k}. \tag{31}$$

Using the setting for the proportional gain, i.e.,

$$K_p = \frac{1}{k(T_c+\tau)} = \frac{T}{K(T_c+\tau)}, \tag{32}$$

where K is the gain and T the time constant in the first order process. Note that the slope is $k = \frac{K}{T}$ in case of an integrating process. Hence we have

$$T_i = 4\zeta^2(T_c+\tau). \tag{33}$$

Putting $\zeta = 1$ gives real roots and a pole polynomial $\pi(s) = (1+\tau_0 s)^2 = \tau_0^2 s^2 + 2\tau_0 s + 1$. Furthermore using the settings $K_p = \frac{1}{k(T_c+\tau)} = \frac{1}{2k\tau}$ gives the SIMC setting $T_i = 4(T_c+\tau) = 8\tau$ when $T_c = \tau$.

Note also that, according to the pole polynomial coefficients, eq. (30), this gives a time constant, $\tau_0 = \frac{1}{2\zeta}T_i = 4\tau$, for the closed loop responses, and that this is 4 times larger than the specified set-point response time constant, $T_c = \tau$, in the SIMC settings. This inconsistence is believed to be due to the neglection of the time delay in the derivation.

Furthermore, from the polynomial coefficients in eq. (30) another strategy could have been to specify the speed of response, τ_0, and then the integral time constant, T_i, and the proportional gain, K_p, expressed as

$$T_i = 2\zeta\tau_0, \; K_p = \frac{T_i}{k\tau_0^2} = \frac{2\zeta}{k\tau_0}, \tag{34}$$

preferably with $\zeta = 1$. Furthermore, we will propose choosing the speed of the response time constant,

$$\tau_0 = c\tau, \tag{35}$$

and to choose the dimensionless parameter, c, to ensure robustness (sensitivity) measure, and an alternative PI controller setting for integrating plus time delay processes is the result. Hence, the SIMC settings may be formulated as

$$T_i = 2c\tau, \; K_p = \frac{2}{kc\tau}. \tag{36}$$

Eq. (36) is obtained by using eq. (35) in eq. (34) with $\zeta = 1$. Notice that using $c = 4$ in eq. (36) gives the SIMC PI settings with $T_c = \tau$, i.e. presented in eq. (25).

However, we will in the next section use this strategy, but instead of neglecting the time delay in the derivation use some common approximation to the time delay, Pade' approximations, and $e^{-\tau s} \approx 1 - \tau s$, etc.

One should also note that from the above analysis and the relationship given by eq. (31) we find that in order to avoid oscillations in the feedback loop we should chose, $\zeta = 1$, and tune the PI controller such that the product of the proportional gain, K_p, and the integral time, T_i, should be

$$K_p T_i = \frac{4}{k}. \tag{37}$$

Eq. (37) may be used to develop a strategy to re-tune an oscillating feedback loop, as presented in Skogestad (2003).

Notice for later use in Sec. 6 that from eq. (37) and the PI setting, eq. (1), that the SIMC method yields a method product parameter, $\bar{c} = \alpha\beta = K_p k T_i = 4$.

Unfortunately, as also pointed out by Haugen (2010), the response of eliminating load disturbances, v, is slow by these settings and the integral time constant, T_i, may be reduced by a factor of two, i.e. by allowing oscillations and requiring, $\zeta = \frac{\sqrt{2}}{2} \approx 0.7$. This gives

$$T_i = 4\tau. \tag{38}$$

This setting gives a Butterworth pole polynomial $\pi(s) = \tau_0^2 s^2 + \sqrt{2}\tau_0 s + 1$ with $\tau_0 = 2\tau$. The corresponding margins for this setting is, a Gain margin $GM = 2.74$ and a Phase margin, $PM = 34.1°$. This setting gives a considerable faster disturbance response, but the margins are believed to be too low in general. Notice, that this gives a method product parameter, $\bar{c} = \alpha\beta = K_p k T_i = 2$. See further relations to the tuning rules deduced in Sec. 6.

A third choice proposed here is to choose $\zeta = \frac{\sqrt{3}}{2}$. This gives the integral time,

$$T_i = 6\tau. \tag{39}$$

Notice that this gives a method product parameter, $\bar{c} = \alpha\beta = K_p k T_i = 3$, and further relations to the tuning rules deduced in Sec. 5.3 and Sec. 6.

These settings, i.e. with $\zeta = 1$, $\zeta = \frac{\sqrt{2}}{2}$ and $\zeta = \frac{\sqrt{3}}{2}$ are listed in Table 1. As expected, and as we see from Table 1, the maximum time delay to time delay ratio, $\frac{d\tau_{\max}}{\tau}$, is reduced when reducing the relative damping coefficient, ζ. As we will see later on Sec. 6, and Example 6.2, the product parameter $\bar{c} = \alpha\beta = K_p k T_i = 3$ may give a tuning with reasonable margins.

Table 1: PI-controller settings for an integrating plus time delay system, $h_p(s) = k\frac{e^{-\tau s}}{s}$, with gain velocity, k, and time delay $\tau \geq 0$. Setting 1 is the Skogestad IMC (SIMC) setting. Settings 2 are suggested by Haugen (2010) and settings 3 are proposed in this paper. Different settings for the relative damping factor, ζ, used in eq. (29), and maximum time delay error, $d\tau_{\max}$, to time delay, τ, ratio, are illustrated. The corresponding gain Margins GM, and sensitivity indices M_s, are also indicated.

	K_p	T_i	ζ	GM	$\frac{d\tau_{\max}}{\tau}$	M_s
1	$\frac{1}{2k\tau}$	$8\ \tau$	1	2.96	1.59	1.70
2	$\frac{1}{2k\tau}$	$4\ \tau$	$\frac{\sqrt{2}}{2}$	2.74	1.08	1.96
3	$\frac{1}{2k\tau}$	$6\ \tau$	$\frac{\sqrt{3}}{2}$	2.89	1.41	1.77

We will in the next section discuss PI controller settings for integrating plus time delay processes in which we use different approximations to the time delay, i.e., an inverse response approximation $e^{-\tau s} \approx 1 - \tau s$, and Pade' approximations etc.

5. Alternative Settings for Integrating Plus Time Delay Process

5.1. Settings by Approximating Time Delay as Inverse Response

Instead of neglecting the time delay as in the derivation of the SIMC PI settings we will in this section deduce an alternative PI controller tuning for the integral time constant T_i, and the proportional gain, K_p.

Let us study the disturbance response in case of a PI controller. We have

$$\begin{aligned}\frac{y}{v}(s) = \frac{h_p}{1 + h_c h_p} &= \frac{k\frac{e^{-\tau s}}{s}}{1 + K_p\frac{1+T_i s}{T_i s}k\frac{e^{-\tau s}}{s}} \\ &= \frac{kse^{-\tau s}}{s^2 + \frac{K_p k}{T_i}(1 + T_i s)e^{-\tau s}}.\end{aligned} \tag{40}$$

Approximating the delay as an inverse response term we get

$$\begin{aligned}\frac{y}{v}(s) &= \frac{ks(1 - \tau s)}{s^2 + \frac{K_p k}{T_i}(1 + T_i s)(1 - \tau s)} \\ &= \frac{T_i}{K_p}\frac{s(1 - \tau s)}{\frac{T_i}{K_p k}s^2 + (1 + T_i s)(1 - \tau s)}.\end{aligned} \tag{41}$$

The poles are given by the roots of the pole polynomial, i.e.,

$$\begin{aligned}\pi(s) &= \frac{T_i}{K_p k}s^2 + (1+T_i s)(1-\tau s) \\ &= T_i\left(\frac{1}{K_p k} - \tau\right)s^2 + (T_i - \tau)s + 1 \\ &= \tau_0^2 s^2 + 2\tau_0\zeta s + 1. \end{aligned} \tag{42}$$

Comparing the coefficients with the standard second order form polynomial we find

$$\begin{aligned}\tau_0^2 &= \frac{T_i}{K_p k} - T_i\tau \\ &= T_i\left(\frac{1}{K_p k} - \tau\right), \end{aligned} \tag{43}$$

and

$$2\tau_0\zeta = T_i - \tau. \tag{44}$$

Let us now prescribe the speed of response, τ_0, for a given relative damping, ζ, where it makes sense to choose $\zeta = 1$. Hence, we have the following PI controller tuning

$$T_i = 2\tau_0 + \tau,\ \ K_p = \frac{T_i}{k(\tau_0^2 + T_i\tau)} = \frac{2\tau_0 + \tau}{k(\tau_0 + \tau)^2}. \tag{45}$$

This PI controller setting eq. (45), as deduced above, is presented in Tyreus and Luyben (1992) eqs. (2-3) of that paper, where it also was suggested to choose $\tau_0 = \sqrt{10}\tau$. The tuning rules in eq. (45) and deduced above, are similar to the IMC PI settings in Table 1 in Chien and Fruehauf (1990).

Furthermore, we here propose to choose the prescribed speed of the response, τ_0, equal to a factor of the time delay, τ, in order to ensure the same robustness properties, approximately constant as a function of the time delay, i.e., we chose $\tau_0 = c\tau$, and c chosen according to e.g. a prescribed maximum time delay error. With this we propose the settings

$$\begin{aligned}T_i &= (2c+1)\tau \\ K_p &= \frac{2c+1}{k\tau(c^2+2c+1)} = \frac{2c+1}{k\tau(c+1)^2}. \end{aligned} \tag{46}$$

Choosing a factor, $c = 2.75$, gives a Gain margin, $GM \approx 3.15$, and a Phase margin, $PM = 44.61°$, maximum time delay error, $d\tau_{\max} = 1.61\tau$ and $M_s = 1.67$. Choosing a factor, $c = 2.6$, gives a Gain margin , $GM \approx 3.04$, and a Phase margin, $PM = 43.41°$, etc. Simulation results show that this controller tuning gives very good robustness margins for integrating plus time delay processes, and faster load disturbance response compared to the SIMC settings. Notice also that both the Gain margin, GM, and the Phase margin, PM, are constant for varying time delay, τ, constant gain, k, and with a prescribed dimensionless parameter, $c = \frac{\tau_0}{\tau}$. Furthermore, the Gain margin, GM, the Phase margin, PM, and the maximum time delay uncertainty, $d\tau_{\max}$, are constant for varying gain velocity (slope), k, and with constant time delay, τ, and for a prescribed dimensionless parameter, $c = \frac{\tau_0}{\tau}$.

Note that an alternative expression of the settings eq. (46) may be found by defining the parameter $\beta = 2c+1$, i.e., as

$$T_i = \beta\tau,\ \ K_p = \frac{4\beta}{k\tau(\beta+1)^2}. \tag{47}$$

For instance a setting $\beta = 6.5$ gives the same setting as eq. (46) with $c = 2.75$. This last variant may be a simpler formulation in case of tuning as a function of β.

Some related discussion and analysis is done in the following. From $4\zeta^2\tau_0^2 = (T_i - \tau)^2$ we find the following 2nd order polynomial for the relationship between T_i and K_p as a function of the relative damping coefficient ζ, i.e.,

$$T_i^2 - \left(4\zeta^2\left(\frac{1}{K_p k} - \tau\right) + 2\tau\right)T_i + \tau^2 = 0. \tag{48}$$

With the setting $K_p = \frac{1}{2k\tau}$ for the proportional gain we obtain $\tau_0^2 = T_i\tau$. Requiring $\zeta = 1$ gives

$$4T_i\tau = (T_i - \tau)^2, \tag{49}$$

and

$$T_i^2 - 6\tau T_i + \tau^2 = 0, \tag{50}$$

with the positive solution

$$T_i = \frac{6+\sqrt{32}}{2}\tau = (3+2\sqrt{2})\tau \approx 6\tau. \tag{51}$$

This gives very good set-point and disturbance responses. Notice that the setting, eq. (51) is approximately the same as the one proposed in eq. (39).

Putting $\zeta = \frac{\sqrt{2}}{2}$ gives

$$T_i = (2+\sqrt{3})\tau \approx 4\tau, \tag{52}$$

which is approximately the same setting as in eq. (38), and is not considered further.

5.2. Settings by Approximating Time Delay with Pade and Balchen Approximation

We will in this section use a standard first order Pade approximation to the time delay, as well as the alternative approximation presented in Balchen (1990).

Let us study the disturbance response in case of a PI controller. We have

$$\frac{y}{v}(s) = \frac{h_p}{1+h_c h_p} = \frac{k\frac{e^{-\tau s}}{s}}{1+K_p\frac{1+T_i s}{T_i s}k\frac{e^{-\tau s}}{s}} = \frac{kse^{-\tau s}}{s^2+\frac{K_p k}{T_i}(1+T_i s)e^{-\tau s}}. \quad (53)$$

The delay is approximated as follows.

$$e^{-\tau s} \approx \frac{1-\bar{\alpha}s}{1+\bar{\alpha}s}, \quad (54)$$

where $\bar{\alpha} = \frac{\tau}{2}$ gives the first order Pade approximation. An alternative is to use the Balchen (1990) approximation, i.e. with, $\bar{\alpha} = \frac{2\tau}{\pi}$. This gives

$$\frac{y}{v}(s) = \frac{ks\frac{1-\bar{\alpha}s}{1+\bar{\alpha}s}}{s^2+\frac{K_p k}{T_i}(1+T_i s)\frac{1-\bar{\alpha}s}{1+\bar{\alpha}s}} = \frac{T_i}{K_p}\frac{s\frac{1-\bar{\alpha}s}{1+\bar{\alpha}s}}{\frac{T_i}{K_p k}s^2+(1+T_i s)\frac{1-\bar{\alpha}s}{1+\bar{\alpha}s}}. \quad (55)$$

which is equivalent with

$$\frac{y}{v}(s) = \frac{T_i}{K_p}\frac{s(1-\bar{\alpha}s)}{\frac{T_i}{K_p k}s^2(1+\bar{\alpha}s)+(1+T_i s)(1-\bar{\alpha}s)}, \quad (56)$$

and

$$\frac{y}{v}(s) = \frac{T_i}{K_p}\frac{s(1-\bar{\alpha}s)}{\frac{\bar{\alpha}T_i}{K_p k}s^3+T_i(\frac{1}{K_p k}-\bar{\alpha})s^2+(T_i-\bar{\alpha})s+1}. \quad (57)$$

Hence, we have the pole polynomial

$$\pi(s) = \bar{\alpha}\frac{T_i}{K_p k}s^3+T_i(\frac{1}{K_p k}-\bar{\alpha})s^2+(T_i-\bar{\alpha})s+1. \quad (58)$$

We may now find a relationship between the controller parameters by specifying the polynomial coefficients. One choice is a Butterworth configuration with $\zeta = \frac{\sqrt{2}}{2}$ in a prescribed 3rd order pole polynomial

$$\pi(s) = (1+\tau_0 s)(\tau_0^2 s^2+2\zeta\tau_0 s+1) = \tau_0^3 s^3+(1+2\zeta)\tau_0^2 s^2+(1+2\zeta)\tau_0 s+1. \quad (59)$$

We will instead for the sake of increased robustness in the resulting feedback system choose, $\zeta = 1$, and three multiple real time constants, i.e. a prescribed pole polynomial

$$\pi(s) = (1+\tau_0 s)(\tau_0^2 s^2+2\tau_0 s+1) = (1+\tau_0 s)^3 = \tau_0^3 s^3+3\tau_0^2 s^2+3\tau_0 s+1. \quad (60)$$

Comparing the coefficients in polynomials (58) and (60) we find

$$\tau_0^3 = \bar{\alpha}\frac{T_i}{K_p k}, \; 3\tau_0^2 = T_i\left(\frac{1}{K_p k}-\bar{\alpha}\right), \; 3\tau_0 = T_i-\bar{\alpha}. \quad (61)$$

This problem is a little bit tricky. In order to use the three coefficients in eq. (61) we first eliminate the ratio, $\frac{T_i}{K_p}$, from the coefficients, $\tau_0^3 = \bar{\alpha}\frac{T_i}{K_p k}$, and, $3\tau_0^2 = T_i(\frac{1}{K_p k}-\bar{\alpha})$, and use the third coefficient, $3\tau_0 = T_i-\bar{\alpha}$, to eliminate T_i, and find the 3rd order polynomial for, τ_0, as

$$\frac{1}{\bar{\alpha}^3}\tau_0^3-\frac{3}{\bar{\alpha}^2}\tau_0^2-\frac{3}{\bar{\alpha}}\tau_0-1 = 0, \quad (62)$$

or equivalently written in terms of the ratio, $\frac{\tau_0}{\bar{\alpha}}$, i.e.,

$$\left(\frac{\tau_0}{\bar{\alpha}}\right)^3-3\left(\frac{\tau_0}{\bar{\alpha}}\right)^2-3\frac{\tau_0}{\bar{\alpha}}-1 = 0. \quad (63)$$

This polynomial has one real root $\lambda = \frac{\tau_0}{\bar{\alpha}}$, which may be analytically expressed as

$$\tau_0 = \overbrace{(2^{\frac{1}{3}}+2^{\frac{2}{3}}+1)}^{\lambda}\,\bar{\alpha} \approx 3.8473\bar{\alpha}. \quad (64)$$

By defining the parameter in the Pade' approximation as $\bar{\alpha} = p\tau$, where $p = \frac{1}{2}$ gives the Pade approximation and $p = \frac{2}{\pi}$ the Balchen approximation, we find that the closed loop time constant is

$$\tau_0 = \lambda p\tau, \quad (65)$$

and the dimensionless parameter is $c = \lambda p$. Hence the integral time is obtained as

$$T_i = 3\tau_0+\bar{\alpha}. \quad (66)$$

Interestingly, from the coefficients in eq. (61), by using that, $3\tau_0^3 = T_i(\frac{1}{K_p k}-\bar{\alpha})\tau_0$, we find the linear expression involving T_i and K_p as,

$$3\bar{\alpha}\frac{1}{K_p k} = \left(\frac{1}{K_p k}-\bar{\alpha}\right)\frac{1}{3}(T_i-\bar{\alpha}). \quad (67)$$

Solving eq. (67) for the proportional gain, gives

$$K_p = \frac{T_i-10\bar{\alpha}}{k(T_i-\bar{\alpha})\bar{\alpha}} = \frac{\tau_0-3\bar{\alpha}}{\bar{\alpha}k\tau_0}, \quad (68)$$

or alternatively from, $\tau_0^3 = \bar{\alpha}\frac{T_i}{K_p k}$, gives

$$K_p = \bar{\alpha}\frac{T_i}{\tau_0^3 k} = \bar{\alpha}\frac{3\tau_0+\bar{\alpha}}{k\tau_0^3}. \quad (69)$$

Note that we have used the expression eq. (66) in eqs. (68-69).

Eqs. (64), (66) and (68) with $\bar{\alpha} = p\tau$ ($p = \frac{1}{2}$ Pade approximation, $p = \frac{2}{\pi}$ Balchen approximation) give PI controller settings in terms of the closed loop time constant, $\tau_0 = c\tau$, given by eq. (64). This may be equivalently formulated in the following Proposition 5.1.

Proposition 5.1 (PI tuning rules: Pade approx.)

Given process parameters, i.e., velocity gain k, and time delay τ. Chose the tuning parameter p, preferably in the range $0.4 \leq p \leq 0.7$, and with $p = 0.5$ (Pade') as default. We have the following PI controller tuning rules.

$$\bar{\alpha} = p\tau, \quad (70)$$
$$\lambda = 2^{\frac{1}{3}} + 2^{\frac{2}{3}} + 1, \quad (71)$$
$$c = \lambda p. \quad (72)$$

We have

$$T_i = (3c + p)\tau = (3\lambda + 1)p\tau, \quad (73)$$

and

$$K_p = \frac{c - 3p}{pck\tau} = \frac{\lambda - 3}{p\lambda k\tau}. \quad (74)$$

From Proposition 5.1 we have the concrete settings

$$T_i = 12.542p\tau = 6.271\tau, \quad K_p = \frac{0.2202}{p}\frac{1}{k\tau} = \frac{0.441}{k\tau} \text{ for } p = \frac{1}{2}, \quad (75)$$

$$T_i = 7.985\tau, \; K_p = \frac{0.3459}{k\tau} \text{ for } p = \frac{2}{\pi}. \quad (76)$$

The PI controller settings in Proposition 5.1 with Pade' approximation $p = 0.5$ gives very good margins, i.e., a gain margin $GM \approx 3.3$, a phase margin $PM = 44.4°$, a maximal time delay error $d\tau_{\max} \approx 1.67\tau$ and and $M_s \approx 1.64$. The corresponding SIMC PI settings with $T_i = 8\tau$ give $GM \approx 2.96$, $PM = 46.86°$, $d\tau_{\max} \approx 1.58\tau$ and $M_s \approx 1.7$. Furthermore, the disturbance response is compared with other settings in Examples 7.2 and 7.3. Varying the tuning parameter p in the range $0.4 \leq p \leq 0.7$ gives an $M_s \approx 1.94$ for $p = 0.4$ and an $M_s \approx 1.32$ for $p = 0.7$. Hence, a large p gives a more conservative tuning.

5.2.1. Other Related Details

Alternatively, we may instead solve for the integral time (from polynomial coefficients in eq. (61) and using eq. (67)) and obtain

$$T_i = \frac{\bar{\alpha}(\frac{10}{K_pk} - \bar{\alpha})}{\frac{1}{K_pk} - \bar{\alpha}} = \frac{1}{\bar{\alpha}}K_pk\tau_0^3. \quad (77)$$

Note also that the integral gain, $\frac{K_p}{T_i}$, in the PI controller may be expressed as

$$\frac{K_p}{T_i} = \frac{1}{k\lambda^3p^2\tau^2}. \quad (78)$$

Notice that a PI controller in the time domain may be expressed as a feedback, $u = K_pe + \frac{K_p}{T_i}z$, where the ratio $\frac{K_p}{T_i}$ is the feedback gain from the integration controller state, $\dot{z} = e$ or $z = \int_0^\infty edt$. See further relations to the Integral Error, $IE = \int_0^\infty edt$ in Åström and Hägglund (1995).

Notice, that the PI controller tuning stated in Proposition 5.1, according to the PI controller parameters, gives $\alpha = \frac{\lambda-3}{p\lambda}$ and $\beta = (3\lambda+1)p$, and the PI controller parameters from eq. (1). Finally, note that the method product parameter, $\bar{c} = \alpha\beta$, is constant and given by

$$\bar{c} = \alpha\beta = \frac{(\lambda - 3)(3\lambda + 1)}{\lambda} \approx 2.7622. \quad (79)$$

The setting in Proposition 5.1 which gives the method parameter $\bar{c}$, in eq. (79) is further discussed in connection with a prescribed maximum time delay error tuning approach in Sec. 6.

5.3. Settings by Using Approximation $e^{-\tau s} \approx \frac{1}{1+\tau s}$

In the model reduction procedure proposed in Skogestad (2001) small time lag constants are approximated with a time delay, i.e., $\frac{1}{1+Ts} \approx e^{-Ts}$ where T is a time constant much smaller to the dominant. Hence, it also makes sense to approximate a (small) time delay with a time constant as in the following.

Finally we will present another PI controller setting for integrating plus time delay systems to those found in Sections 5.1 and 5.2. Using the approximation

$$e^{-\tau s} = \frac{1}{e^{\tau s}} \approx \frac{1}{1 + \tau s}, \quad (80)$$

in the disturbance response transfer function eq. (55) gives

$$\frac{y}{v}(s) = \frac{h_p}{1 + h_ch_p} = \frac{ks\frac{1}{1+\tau s}}{s^2 + \frac{K_pk}{T_i}(1 + T_is)\frac{1}{1+\tau s}} = \frac{T_i}{K_p}\frac{s\frac{1}{1+\tau s}}{\frac{T_i}{K_pk}s^2 + (1 + T_is)\frac{1}{1+\tau s}}. \quad (81)$$

From the denominator in eq. (81) we find the pole polynomial

$$\pi(s) = \frac{T_i\tau}{K_pk}s^3 + \frac{T_i}{K_pk}s^2 + T_is + 1. \quad (82)$$

Requiring real poles and $\zeta = 1$ as in eqs. (59) and (60) and comparing coefficients gives

$$\tau_0^3 = \frac{T_i\tau}{K_pk}, \; 3\tau_0^2 = \frac{T_i}{K_pk}, \; 3\tau_0 = T_i. \quad (83)$$

From this we find the closed loop time constant as

$$\tau_0 = 3\tau, \tag{84}$$

and the integral time

$$T_i = 3\tau_0 = 9\tau, \tag{85}$$

and the proportional gain

$$K_p = \frac{1}{\tau_0 k} = \frac{1}{3k\tau}. \tag{86}$$

This results in a rather conservative setting with good margins, i.e., Gain margin, $GM \approx 4.46$, Phase margin, $PM \approx 52.33°$, a maximum time delay error $d\tau_{\max} \approx 2.61\tau$ and a sensitivity index $M_s \approx 1.42$. The set-point and disturbance responses may be rather slow by this setting, but a rather safe setting indeed. Notice that the tuning rules deduced above give a method product parameter, $\bar{c} = \alpha\beta = K_p k T_i = 3$, and relations to the tuning rules deduced in the next Sec. 6.

6. Tuning for Maximum Time Delay Error

In order to get some insight into the Phase margin, PM, of the closed loop system and the maximum time delay error, $d\tau_{\max}$, we work out some analytic results in the following, which lead to some interesting results.

Consider an integrator plus time delay system, $h_p(s) = k\frac{e^{-\tau s}}{s}$, where k is the Gain velocity and τ the time delay, and a PI controller. The loop transfer function, $h_0(s) = h_c(s)h_p(s)$, is

$$h_0(s) = K_p\frac{1+T_i s}{T_i s}k\frac{e^{-\tau s}}{s}. \tag{87}$$

The frequency response is given by, $h_0(j\omega) = |h_0(j\omega)|e^{j\angle h_0(j\omega)}$, where the magnitude is given by

$$|h_0(j\omega)| = \frac{K_p k}{T_i \omega^2}\sqrt{1+(T_i\omega)^2}, \tag{88}$$

and the phase angle as

$$\angle h_0(j\omega) = -\tau\omega - \pi + \arctan(T_i\omega). \tag{89}$$

First we find the Gain crossover frequency, ω_c, analytically such that $|h_0(j\omega)_c| = 1$. From this we find analytic results for the Phase margin, $PM = \angle h_0(j\omega_c)+\pi$, and the maximum time delay error, $d\tau_{\max}$, such that, $0 = PM - d\tau_{\max}\omega_c$, in the following.

Define a factor, f, as

$$f = \frac{1+\sqrt{1+\frac{4}{(K_pT_ik)^2}}}{2}. \tag{90}$$

The Gain crossover frequency is analytically given by

$$\omega_c = \sqrt{f}K_p k. \tag{91}$$

A proof of eq. (91) is given in Appendix B. Let us use the defined expressions for the PI controller parameters as in eq. (1), and we find

$$f = \frac{1+\sqrt{1+\frac{4}{(\alpha\beta)^2}}}{2}. \tag{92}$$

The Gain crossover frequency is then given by

$$\omega_c = \sqrt{f}\frac{\alpha}{\tau}. \tag{93}$$

We find the Phase margin in radians, analytically as

$$PM = -\sqrt{f}\alpha + \arctan(\sqrt{f}\alpha\beta), \tag{94}$$

and the maximum time delay error analytically as

$$d\tau_{\max} = \frac{PM}{\omega_c} = \delta\tau. \tag{95}$$

where coefficient, δ, is defined as

$$\delta = \frac{-\sqrt{f}\alpha + \arctan(\sqrt{f}\alpha\beta)}{\sqrt{f}\alpha} = \frac{\arctan(\sqrt{f}\alpha\beta)}{\sqrt{f}\alpha} - 1. \tag{96}$$

We find that the maximum time delay error, $d\tau_{\max}$, is proportional with the time delay, τ, with proportional coefficient, δ, defined above.

The above states that the ratio, $\frac{d\tau_{\max}}{\tau} = \delta$ is a function of the PI controller parameters β and α in eq. (1), i.e., $\delta = f(\alpha, \beta)$.

Consider now the case in which the product, $\bar{c} = \alpha\beta$, is constant, then eq. (96) may be written as

$$\delta = a\frac{1}{\alpha} - 1, \tag{97}$$

and

$$\delta = \frac{a}{\bar{c}}\beta - 1, \tag{98}$$

where the parameter, a, given by

$$a = \frac{\arctan(\sqrt{f}\alpha\beta)}{\sqrt{f}}, \tag{99}$$

is a function of $\bar{c} = \alpha\beta$ and constant. Notice that the parameter, f, is defined by eq. (92).

We have the following algorithm.

Algorithm 6.1 (Max time delay error tuning)

Define the method product parameter

$$\bar{c} = \alpha\beta. \tag{100}$$

From this we may express, β, as a linear function of a prescribed $\delta > 0$, in order to ensure stability of the feedback system. We have

$$\beta = \frac{\bar{c}}{a}(\delta + 1), \tag{101}$$

where parameter, a, is defined in eq. (99). Notice that, α, then is found as

$$\alpha = \frac{\bar{c}}{\beta} = \frac{a}{\delta + 1}. \tag{102}$$

Or equivalently in terms of the PI controller parameters

$$T_i = \frac{\bar{c}}{a}(\delta + 1)\,\tau, \tag{103}$$

$$K_p = \frac{a}{k\tau(\delta + 1)}. \tag{104}$$

This is a useful result. Algorithm 6.1 and eqs. (101) and (102), may be used in connections with methods in which the product $\bar{c} = \alpha\beta$ is constant, in order to find the PI controller parameters $T_i = \beta\tau$ and $K_p = \frac{\alpha}{k\tau}$ such that the closed loop system has a prescribed maximum time delay error ratio, $\delta = \frac{d\tau_{\max}}{\tau}$. Or, in other words the above states that the PI controller parameters, α and β may be expressed in terms of the method dependent product parameter, $\bar{c} = \alpha\beta$, and the maximum time delay error, $d\tau_{\max}$, to time delay, τ, ratio parameter δ.

Before continuing, we illustrate the above algorithm in an Example, in order to improve the robustness in the original ZN tuning in eq. (10).

Example 6.1 (ZN with increased margins) *Consider the original ZN tuning in eq. (10) in which $\alpha = \frac{\pi}{4.4} \approx 0.714$, $\beta = \frac{4}{1.2} \approx 3.33$. The maximum time delay error for the original ZN tuning is $\frac{d\tau_{max}}{\tau} = \delta \approx 0.562$ and the sensitivity index $M_s \approx 2.864$.*

For the original ZN method we have the product, $\bar{c} = \alpha\beta \approx 2.38$. Specifying a maximum time delay error parameter, $\delta = \frac{d\tau_{max}}{\tau} = 1.6$. Using eqs. (101) and (102) gives modified ZN PI controller parameters

$$\alpha = 0.4209, \quad \beta = 5.5471. \tag{105}$$

This modified PI controller ZN tuning, $K_p = \frac{\alpha}{k\tau}$ and $T_i = \beta\tau$, for an integrating plus time delay process has margins $GM = 3.3455$, sensitivity index $M_s = 1.6568$ and prescribed $\frac{d\tau_{max}}{\tau} = 1.6$. This modified ZN tuning has relatively smooth closed loop responses with a relative damping slightly less than one. The ZN method parameter $\bar{c} = 2.38$ is relatively low but see later discussions.

A second motivating example is presented in the following.

Example 6.2 (Tuning with reduced margins) *Consider the tuning deduced in Sec. 5.3 where we obtained PI controller parameters $T_i = 9\tau$ and $K_p = \frac{1}{3k\tau}$, with as we believe, in general too conservative margins. However, the product parameter seems acceptable, i.e. $\bar{c} = \alpha\beta = 3$. Specifying a maximum time delay error parameter, $\delta = \frac{d\tau_{max}}{\tau} = 1.6$. Using eqs. (101) and (102) gives modified PI controller parameters*

$$\alpha = 0.4630, \quad \beta = 6.4789. \tag{106}$$

This modified PI controller, $K_p = \frac{\alpha}{k\tau}$ and $T_i = \beta\tau$, for an integrating plus time delay process has gain margin $GM = 3.147$, sensitivity index $M_s = 1.674$ and prescribed $\frac{d\tau_{max}}{\tau} = 1.6$.

Probably, the most important with a PI controller setting, is the robustness against model uncertainty, in connection to reasonable fast and smooth closed loop set-point and disturbance responses. A maximum time delay error of about, $d\tau_{\max} = 1.6\tau$, seems reasonable. This is approximately, equal to the maximum time delay error for the SIMC setting, $d\tau_{\max} = 1.59\tau$.

One idea, is to find theoretical arguments for setting the product parameter, $\bar{c}$, such that the closed loop system obtains some optimal settings, e.g. minimize M_s for a prescribed δ.

Using the tuning rules deduced in Proposition 5.1 which gives the product parameter as in eq. (79), i.e. $\bar{c} = 2.7622$. Choosing a prescribed maximum time delay error ratio parameter, $\delta = 1.75$ gives a corresponding Gain margin, $GM = 3.4148$ and smooth responses approximately as fast as the corresponding responses by using the SIMC setting, on an integrator plus time delay example with gain velocity, $k = 1$, and time delay, $\tau = 1$.

The method parameter, $\bar{c} = \alpha\beta$, may be viewed as a tuning parameter. SIMC are using a product parameter, $\bar{c} = 4$, and the corresponding Gain margin is as low as, $GM \approx 2.96$, but the maximum time delay error seems acceptable, i.e., $d\tau_{\max} = 1.59\tau$. This setting gives a relatively slow disturbance rejection, as commented upon in Sec. 4.2 and Example 7.1, see also Haugen (2010). Hence, we may view $\bar{c} = 4$ as an upper limit for this parameter. Simulation experiments show that a lower limit before oscillations (a relative damping less than one) is approximately $\bar{c} = 2.4$, (on an integrator model with gain velocity $k = 1$ and time delay $\tau = 1$). Based on the investigations in this paper we propose a relatively wide range for the method product parameter, $\bar{c}$, to be chosen according to

$$1.5 \leq \bar{c} \leq 4. \tag{107}$$

Notice that the tuning in Proposition 5.1 with $p = 0.5$ results in the parameters $\bar{c} = 2.76$ and $\delta = 1.67$.

For a prescribed δ, we find that a method product parameter $\bar{c} = 2.0$ is optimal in the sense that M_s is minimized. This is found from simulation experiments and illustrated in Figure 1.

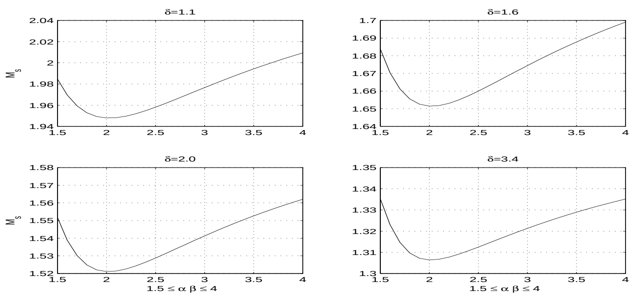

Figure 1: PI control of integrating plus delay process, $h_p(s) = k\frac{e^{-\tau s}}{s}$. PI-controller $h_c(s) = K_p\frac{1+T_i s}{T_i s}$ with settings $k = 1$ and $\tau = 1$. Sensitivity index M_s as a function of varying method product parameter $1.5 \leq \bar{c} \leq 4$. and varying δ. $\bar{c} = 2.0$ is optimal in the sense that M_s is minimized.

Furthermore we propose to choose the maximum time delay error ratio $\delta > 0$ in order to ensure stability, and choosing δ in the range

$$1.1 \leq \delta \leq 3.4, \tag{108}$$

for robustness and in order to ensure $1.3 \leq M_s \leq 2.0$, seems reasonable (Åström and Hägglund (1995) p. 125). See Figure 1 for illustration. Notice also that the Phase Margin is in the range $31.4^\circ \leq PM \leq 59^\circ$ when choosing the tuning parameters $\bar{c}$ and δ from eqs. (107) and (108). See also eq. (129) in the following discussions in Sec. 8.3 for an analytic expression for the phase margin in connection with Algorithm 6.1.

A MATLAB m-file function for the main result in Algorithm 6.1 is enclosed in Appendix C.

Furthermore, parameters β and α are often functions of the closed loop time constant tuning parameter, τ_0, which may be expressed proportional to the time delay, i.e., $\tau_0 = c\tau$. Interestingly, we find that the maximum time delay error ratio parameter, δ, given by the analytic expression eq. (96) is linear in the parameter, c. This holds for methods in which the product $\alpha\beta$ is constant. It also turns out that the parameter, β, is linear in $\frac{d\tau\text{max}}{\tau}$. This is discussed in the following Subsections.

6.1. SIMC Formulation in Terms of Prescribed Maximum Time Delay Error

As an illustrative example we use the SIMC tuning rules for a pure integrating plus time delay process with $K_p = \frac{1}{k(T_c+\tau)}$ and $T_i = 4(T_c + \tau)$ with $T_c = c\tau$ and $c = 1$. Hence parameters $\alpha = 0.5$, $\beta = 8$ and method parameter $\bar{c} = \alpha\beta = 4$. This gives the maximum time delay error relative to the time delay from eq. (96), analytically as

$$\begin{aligned}
\frac{d\tau\text{max}}{\tau} &= \delta \\
&= \frac{4\arctan(2\sqrt{2+\sqrt{5}}) - \sqrt{2+\sqrt{5}}}{\sqrt{2+\sqrt{5}}} \\
&= \frac{4}{\sqrt{2+\sqrt{5}}}\arctan(2\sqrt{2+\sqrt{5}}) - 1 \\
&\approx 1.59.
\end{aligned} \tag{109}$$

Equivalently, the result eq. (109) may be found from eq. (97).

For a chosen prescribed time constant, $T_c \geq \tau$, of the set-point response and a time delay, τ, then for the SIMC method we have the PI-controller parameters in, eq. (1) as follows,

$$c = \frac{T_c}{\tau} \tag{110}$$

$$\beta = 4(c+1), \quad \alpha = \frac{1}{c+1}. \tag{111}$$

The maximum time delay error may be expressed as follows,

$$\begin{aligned}
\frac{d\tau\text{max}}{\tau} = \delta &= ac + a - 1 \\
&= 1.2948c + 0.2948,
\end{aligned} \tag{112}$$

where the parameter, a, is defined in (99). Expression eq. (112) is obtained from the analytic relationship eq. (97). Notice that since the product parameter, $\bar{c} = \alpha\beta = 4$, then eq. (97) with eq. (112), may be written as

$$c = \frac{1}{a}(\delta + 1 - a), \tag{113}$$

where the parameter, $a = 1.2948$.

Using Algorithm 6.1 we obtain the following. The parameter β in the integral time, $T_i = \beta\tau$, is linear in the maximum time delay ratio, and found to be

$$\beta = 3.089(\delta + 1). \tag{114}$$

where $\delta = \frac{d\tau\text{max}}{\tau}$. Furthermore we find the proportional gain

$$\alpha = \frac{4}{\beta} = \frac{1.2948}{\delta + 1}. \tag{115}$$

Here we have modified the SIMC tuning rules such that the PI controller parameters are found in terms of a prescribed maximum time delay error. Notice that the SIMC method gives a prescribed time delay error, $\delta = 1.59$ as shown in eq. (109). Using $\delta = 1.59$ in eqs. (114), (115) and (113) gives the SIMC tuning rules, i.e. with $T_c = \tau$ and $c = 1$ from eq. (113). With the above we may find new PI controller parameters in terms of a prescribed maximum time delay error ratio, $\delta = \frac{d\tau_{\max}}{\tau}$, instead of the closed loop time constant, T_c.

From the above we have found the relationship

$$T_c = \overbrace{\left(\frac{\delta + 1}{a} - 1\right)}^{c} \tau, \tag{116}$$

were $a = 1.2984$ is constant and only a function of the method product parameter $\bar{c} = 4$, for the SIMC method. The interpretation of eq. (116) is that the SIMC tuning rules may be expressed in terms of the maximum time delay error ratio parameter, δ, as the tuning parameter instead of the closed loop time constant, T_c. Furthermore we also find from eq. (116), that in order to ensure stability of the feedback system ($\delta > 0$) we have to choose the SIMC tuning parameter in the range, $(\frac{1}{a} - 1)\tau < T_c < \infty$. Hence, we have reduced the range for the SIMC tuning parameter, Skogestad (2001, 2003, 2004), where the range is specified as $-\tau \leq T_c \leq \infty$ in order to ensure a positive and nonzero controller gain.

6.2. Alternative IMC Formulation

For a chosen prescribed time constant, τ_0, of the set-point response and a time delay, τ, then for the IMC method we have the PI-controller parameters in, eq. (1) as follows,

$$c = \frac{\tau_0}{\tau} \tag{117}$$

$$\beta = 2c + 1, \quad \alpha = \frac{2c + 1}{(c + 1)^2}, \tag{118}$$

For this method we find that the product, $\bar{c} = \alpha\beta$, is not constant and given by

$$\bar{c} = \alpha\beta = \frac{(2c + 1)^2}{(c + 1)^2} = \frac{4\beta^2}{(\beta + 1)^2}. \tag{119}$$

Notice that using a fixed parameter, $c = 2.75$, gives a product parameter $\bar{c} \approx 3.00$ and tuning rules almost similar to that in Example 6.2 is the result, e.g. with a Gain margin, $GM \approx \pi$ and a maximum time delay error, $d\tau_{\max} = 1.61\tau$.

The maximum time delay error may be approximately expressed as follows,

$$\frac{d\tau_{\max}}{\tau} = \delta = 0.6488c - 0.1803. \tag{120}$$

Expression eq. (120) was found by linear regression and the error over the range $1 \leq c \leq 10$ measured with the Frobenius norm is about 0.03. The coefficient, c, related to the closed loop time constant, $\tau_0 = c\tau$, is then given by

$$c = 1.5413\delta + 0.2779. \tag{121}$$

The parameter, β, in the integral time, $T_i = \beta\tau$, is approximately linear in the maximum time delay ratio, and found to be

$$\beta = 3.0827\delta + 1.5557. \tag{122}$$

where, $\delta = \frac{d\tau_{\max}}{\tau}$. Furthermore we find

$$\alpha = \frac{4\beta}{(\beta + 1)^2}. \tag{123}$$

Here we have presented a variant of the IMC tuning rules, (45), such that the PI controller parameters are found in terms of a prescribed maximum time delay error ratio parameter, $\delta = \frac{d\tau_{\max}}{\tau}$, instead of the closed loop time constant, τ_0.

7. Simulation Examples

In order to compare different controller settings against each other we will in the examples use the same indices as defined in Skogestad (2003), Skogestad (2004). See also Åström and Hägglund (1995) and Seborg et al. (1989) for such indices.

To evaluate the output from set-point and disturbance responses we use the Integrated Absolute Error (IAE) index, i.e.,

$$IAE = \int_0^\infty |e| dt, \tag{124}$$

where, $e = r - y$, is the control deviation error and, r, the reference.

Notice, that the IAE may be calculated recursively in discrete time as, $IAE_{k+1} = IAE_k + \Delta t|e_k|$, where, Δt, is the sampling time, and, k, discrete time.

To evaluate the amount of input used we use the Total value (TV) index formulated in discrete time as

$$TV = \sum_{k=1}^{\infty} |\Delta u_k|, \tag{125}$$

where, $\Delta u_k = u_k - u_{k-1}$, is the control rate of change.

In the upcoming examples we evaluate the IAE and TV index, eq. (124) and (125), respectively, for the entire simulation interval, $0 \leq t \leq t_{\text{final}}$, i.e., both set-point and disturbance responses are contained and measured in the IAE and TV index values presented in the examples. As default we are using a positive unit step change in the reference, r, at time $t = 0$, and a unit positive step change in the input disturbance, v, at time $t = \frac{t_{\text{final}}}{2}$.

As the default we will in the examples compare with the SIMC tuning rule for fast response with good margins, $T_c = \tau$, Skogestad (2003).

Example 7.1 (Settings in Table 1)
Given an integrator plus time delay system described by the transfer function

$$h_p(s) = k\frac{e^{-\tau s}}{s}, \tag{126}$$

with gain velocity $k = 1$ and time delay $\tau = 1$. The results by using a PI controller with settings as in Table 1 are illustrated in Figure 2, which shows set-point and disturbance rejection responses after a unit step in the reference, $r = 1$ at time $t = 0$, and a unit step in the disturbance from $v = 0$ to $v = 1$ at time $t = 40$. As we see the SIMC settings give a relatively slow response from both the set-point and the disturbance. The Butterworth setting (2) results in the fastest responses but has small oscillations. The settings derived in this paper (3) result in nice, smooth response approximately as fast as the response of the Butterworth settings.

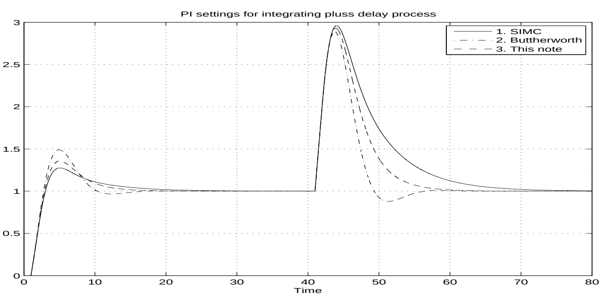

Figure 2: PI control of integrating plus delay process, $h_p(s) = k\frac{e^{-\tau s}}{s}$. PI-controller $h_c(s) = K_p\frac{1+T_i s}{T_i s}$ with settings as in Table 1.

Example 7.2 (Proposition 5.1 and eq. (46))
We will in this example consider the same integrator example as in Example 7.1 and we will compare the different PI controller settings presented in Sec. 5, with the SIMC setting Skogestad (2001), the TL setting in Tyreus and Luyben (1992) (see also eq. (9) in this paper) and the IMC PI controller parameters as in Chien and Fruehauf (1990) (see also eq. (45)).

The PI controller settings from Sec. 5 which are considered are: 1) The modified IMC setting, eq. (46) for different different closed loop time constant, $\tau_0 = c\tau$, i.e., different parameters, c. 2) The PI controller setting in Proposition 5.1 with $p = 0.5$. Some results are presented in Table 2.

Table 2: PI-controller settings for an integrating plus time delay process $h_p(s) = k\frac{e^{-\tau s}}{s}$ with gain, $k = 1$, and time delay $\tau = 1$. Comparing SIMC setting in eq. (25), TL settings in eq. (9), IMC setting Chien and Fruehauf (1990) with $\tau_0 = \sqrt{10}\tau$ as in eq. (45), the PI controller setting in eq. (46) with $c = 2.75$, etc., and the setting proposed in Proposition 5.1 with $p = 0.5$.

	K_p	T_i	GM	PM	$d\tau_{\max}$	IAE	TV
SIMC	0.5	8	2.96	46.86	1.59	19.91	2.28
TL	0.49	8.8	3.06	48.54	1.69	21.99	2.18
IMC	0.423	7.3	3.48	47.50	1.87	21.61	2.17
c=2.75	0.46	6.5	3.15	44.61	1.61	18.12	2.34
c=2.85	0.45	6.7	3.23	45.34	1.67	18.93	2.29
c=2.95	0.44	6.9	3.31	46.08	1.74	19.77	2.25
p=0.5	0.441	6.3	3.3	44.42	1.67	18.39	2.33

Example 7.3 (Algorithm 6.1)
The same integrator example as in Example 7.1 is considered. We will in this section illustrate different PI controller settings obtained by using Algorithm 6.1 in Sec. 6. In Table 3 the setting with product parameter $\bar{c} = 2.76$ is from Proposition 5.1 with $p = 0.5$ which results in a maximum time delay error ratio $\delta = 1.67$. The theoretical background for the parameter $\bar{c} = 3$ is from the tuning in Sec. 5.3 as well as the discussion in Sec. 4.2. The maximum time delay error ratio, $\delta = \frac{d\tau_{max}}{\tau} = 1.6$ is prescribed. Simulation results are illustrated in Figure 3.

Furthermore in Table 4 we illustrate the different settings obtained from Algorithm 6.1 with varying method parameter, $\bar{c}$, and where the maximum time delay error ratio is constant, $\delta = 1.59$. As we see from Table 4 decreasing $\bar{c}$ will decrease both the integral time, T_i, and the proportional gain, K_p. At the same time we see that the gain margin increases, the IAE decreases but seems to have a minimum, at the cost of a slightly increased TV. Notice that the IAE seems to obtain a minimum for about $\bar{c} = 2.5$. We also see that the sensitivity index M_s is minimized for $\bar{c} = 2$. This is in agreement with the results in Figure 1.

Notice that reducing the tuning parameter in the SIMC method below τ, i.e. choosing $T_c < \tau$ will decrease T_i but the proportional gain K_p, will increase, and the maximum time delay error, the gain and phase *margins, will in general be reduced. However, notice that at the same time, the IAE decreases and the TV increases. This is illustrated in Table 5. The corresponding sensitivity indices in Table 5 are $M_s = 1.59$, $M_s = 1.88$, $M_s = 2.18$ and $M_s = 2.31$, and the tunings lack robustness when $T_c < 0.75\tau$.*

Table 3: PI-controller settings for an integrating plus time delay process $h_p(s) = k\frac{e^{-\tau s}}{s}$ with gain, $k = 1$, and time delay $\tau = 1$. Comparing SIMC setting in eq. (25), with different settings obtained from Algorithm 6.1. The tuning with method product parameter, $\bar{c} = 2.76$, is from Proposition 5.1 with $p = 0.5$ and where the resulting maximum time delay error ratio is $\delta = 1.67$.

	K_p	T_i	GM	PM	δ	IAE	TV
1) SIMC	0.5	8	2.96	46.86	1.59	19.91	2.28
$\bar{c} = 2.76$	0.441	6.27	3.3	44.42	1.67	18.39	2.33
2) $\bar{c} = 2.76$	0.452	6.12	3.21	43.75	1.60	17.64	2.37
$\bar{c} = 3$	0.463	6.48	3.148	44.54	1.60	18.04	2.35

Table 4: PI-controller settings for an integrating plus time delay process $h_p(s) = k\frac{e^{-\tau s}}{s}$ with gain, $k = 1$, and time delay $\tau = 1$. Illustrating the different settings obtained from Algorithm 6.1 with varying method parameter, $\bar{c}$, and where the maximum time delay error ratio, $\delta = 1.59$, is held constant. Notice that M_s achieves a minimum for $\bar{c} = 2.0$. Notice that the results with $\bar{c} = 4$ are identical to the SIMC tuning rules with $T_c = \tau$.

Alg. 6.1	K_p	T_i	GM	PM	M_s	IAE	TV
$\bar{c} = 4$	0.5	8	2.96	46.86	1.70	19.91	2.28
$\bar{c} = 3$	0.46	6.45	3.13	44.43	1.68	17.93	2.36
$\bar{c} = 2.76$	0.45	6.09	3.2	43.63	1.67	17.53	2.38
$\bar{c} = 2.5$	0.44	5.70	3.28	42.64	1.66	17.25	2.41
$\bar{c} = 2.0$	0.40	4.98	3.52	40.23	1.656	18.10	2.52
$\bar{c} = 1.5$	0.35	4.28	3.96	36.83	1.69	21.22	2.71

Example 7.4 (Lag dominant system)
In Haugen (2010) an experimental setup of an air heater was investigated and it was found that a time constant plus time delay model

$$h_p(s) = K\frac{e^{-\tau s}}{1 + Ts}, \tag{127}$$

with process gain $K = 5.7$, time constant $T = 60$ and time delay $\tau = 4$, approximates the process reasonable well. We here approximate the first order time delay model eq. (127) as in integrating plus time delay process

$$h_p(s) \approx k\frac{e^{-\tau s}}{s}, \tag{128}$$

Table 5: PI-controller settings for an integrating plus time delay process $h_p(s) = k\frac{e^{-\tau s}}{s}$ with gain, $k = 1$, and time delay $\tau = 1$. Illustrating the SIMC settings for different tuning parameters, T_c.

SIMC	K_p	T_i	GM	PM	δ	IAE	TV
$T_c = 1.25\tau$	0.44	9.0	3.36	50.14	1.91	24.41	2.07
$T_c = \tau$	0.50	8.0	2.96	46.86	1.59	19.91	2.28
$T_c = 0.75\tau$	0.57	7.0	2.57	42.65	1.27	15.91	2.62
$T_c = 0.5\tau$	0.67	6.0	2.17	37.04	0.94	12.41	3.25
$T_c = 0.425\tau$	0.70	5.7	2.05	34.97	0.85	11.46	3.57

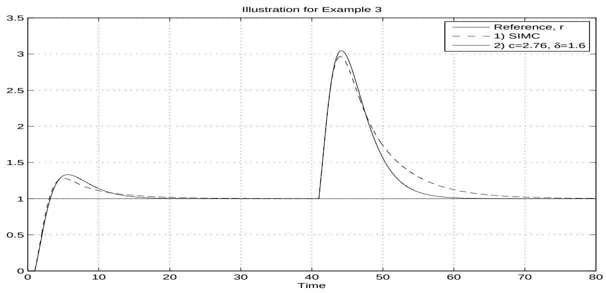

Figure 3: PI control of integrating plus delay process, $h_p(s) = k\frac{e^{-\tau s}}{s}$. PI-controller $h_c(s) = K_p\frac{1+T_i s}{T_i s}$ with settings as in Table 3.

where the gain velocity (slope) is $k = \frac{K}{T} = 0.095$.

The SIMC tuning rules with $T_c = \tau$ give PI controller parameters $K_p = 1.316$ and $T_i = 32$, irrespective of which of the models eqs. (127) and (128) are used.

The SIMC tuning rules are compared with the tuning rules presented in this paper, eq. (46) with $c = 2.75$, the tuning rules in Proposition 5.1 with $p = 0.5$, and the tuning rules in Algorithm 6.1 with $\bar{c} = 2.762$ and prescribed maximum time delay error ratio, $\delta = 1.6$. Results from simulation experiments are illustrated in Table 7.

8. Discussion

8.1. Remarks to Sec. 5.1

The tuning rules in eq. (45) were deduced in Sec. 5.1 by using the approximation $e^{\tau s} \approx 1 - \tau s$. This is identical to the IMC PI settings in Table 1 in Chien and Fruehauf (1990)), but were not deduced in that paper. The tuning rules, eq. (45) are a function of the closed loop time constant, τ_0.

It is in practice usually not trivial to prescribe the closed loop time constant, τ_0, and some trial and error

Table 6: PI-controller settings for an lag dominant first order plus time delay process $h_p(s) = K\frac{e^{-\tau s}}{1+Ts}$ with gain, $K = 5.7$, time constant $T = 60$ and time delay $\tau = 4$. Comparing SIMC setting against the PI controller setting in eq. (46) with $c = 2.75$, Proposition 5.1 with $p = 0.5$ and Algorithm 6.1 with $\tilde{c} = 2.762$.

	K_p	T_i	GM	PM	$d\tau\text{max}$	IAE	TV
SIMC	1.32	32.0	3.06	54.4	7.5	34.3	2.86
c=2.75	1.22	26.0	3.26	52.6	7.6	33.7	2.85
p=0.5	1.16	25.1	3.41	52.7	8	34.5	2.78
Alg. 6.1	1.19	24.5	3.32	51.9	7.6	33.7	2.87

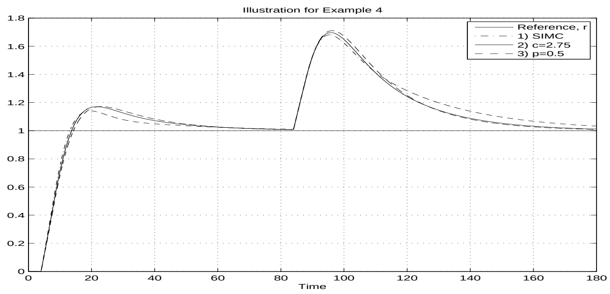

Figure 4: PI control of time lag dominant plus time delay process, $h_p(s) = K\frac{e^{-\tau s}}{1+Ts}$. PI-controller $h_c(s) = K_p\frac{1+T_i s}{T_i s}$ with settings as in Table 6.

procedure is usually necessary. This is also commented upon in Tyreus and Luyben (1992). We propose to choose $\tau_0 = c\tau$ where c is a dimensionless constant, chosen according to some robustness margin, e.g., to a prescribed maximum time delay error, and we propose two new variants in eqs. (46) and (47). The PI-controller tuning variant in eq. (47) is similar to the one in eq. (46) but has some advantages when one wants to tune the controller. For instance, by increasing β in eq. (47) we see that T_i increases and K_p decreases. The same occurs with the setting in eq. (46), but not so simple to see due to the more complicated functions of the parameter, c. Note also, that in case of oscillations in a feedback loop, the correct strategy is to decrease the proportional gain, K_p, and increase the integral time, T_i.

8.2. Remarks to Sec. 5.2

In Sec. 5.2 we used the Pade approximation to the time delay and derived analytically the tuning rules proposed in Proposition 5.1. This method proposes parameters $\alpha = \frac{\lambda-3}{p\lambda}$ and $\beta = (3\lambda+1)p$ where the parameter λ is constant and found analytically as the solution of a 3rd order polynomial. The parameter p is the tuning parameter. Choosing $p = 0.5$ gives the Pade approximation, $p = \frac{2}{\pi}$, gives the Balchen approximation, etc. Notice that the method product parameter, $\bar{c}$, is constant and given by eq. (79).

8.3. Remarks to Sec. 6

Probably the main results of this paper are presented in Sec. 6 and Algorithm 6.1. We found that PI controller tuning rules based on an integrating plus time delay model may be expressed in terms of a method tuning parameter, $\bar{c}$, and a prescribed maximum time delay error ratio parameter, δ. This method has in principle 2 tuning parameters, the method parameter, $\bar{c} = \alpha\beta$, and the maximum time delay error ratio, $\delta = \frac{d\tau\text{max}}{\tau}$. Notice that this method may be used to re-tune existing tuning rules such that the closed loop system obtains a prescribed maximum time delay error ratio, δ. See also discussions in Sec. 6.

From the results in Sec. 6 we find the expression

$$PM = \delta\sqrt{f}\alpha, \tag{129}$$

for the phase margin in radians. Hence, prescribing a maximum time delay error ratio, δ, ensures a prescribed phase margin because f is constant in the method parameter $\bar{c}$.

9. Concluding Remarks

Efficient PI controller tuning rules for integrator plus time delay systems as well as time lag dominant processes, are deduced and presented. The IMC tuning rules are derived by approximating the time delay as an inverse response. We have in this paper derived alternative PI controller tuning rules by using different approximations to the time delay. See Sec. 5.2 and Proposition 5.1.

An algorithm for PI controller tuning of the parameters α and β in the integral time, $T_i = \beta\tau$, and the proportional gain, $K_p = \frac{\alpha}{k\tau}$, is presented. In this method the parameters α and β are functions of a method product parameter, $\bar{c}$, and a prescribed maximum time delay error, $d\tau\text{max}$, to time delay, τ, ratio $\delta = \frac{d\tau\text{max}}{\tau}$. We propose the setting, $\beta = \frac{\bar{c}}{a}(\delta+1)$, and, $\alpha = \frac{a}{\delta+1}$, where the parameter, a, is constant in the method product parameter, $\bar{c} = \alpha\beta$. Based on the investigations in this paper we propose a relatively wide range of the method product parameter, $\bar{c}$, to be chosen according to $1.5 \leq \bar{c} \leq 4$. Prescribing the maximum time delay error ratio $\delta > 0$ in order to ensure stability, and choosing δ in the range $1.1 \leq \delta \leq 3.4$ for robustness seems reasonable. Furthermore, for a prescribed δ we found that the sensitivity index M_s is minimized for $\bar{c} = 2.0$. This method has two tuning parameters $\bar{c}$ and δ, which gives flexibility for tuning. Furthermore, the

PI controller parameters, α, and, β, obtained from the Algorithm 6.1 presented in this paper, are independent of the model parameters.

Some theoretical justifications for the possibility of improving the load disturbance in the closed loop time response, for integrating plus time delay processes, in the SIMC method, are given. Furthermore, the possibility of using the maximum time delay error ratio δ as a tuning parameter in the SIMC tuning rules, instead of the closed loop time constant is proposed, in eq. (116).

Derivation of the IMC PI controller tuning rules, eq. (45), for an integrating plus time delay process are presented. From this, some alternative PI controller tuning rules are presented in eqs. (46) and (47), i.e., by using that the closed loop response time constant may be expressed as, $\tau_c = c\tau$, for some dimensionless parameter c, and this parameter may be chosen to ensure some robustness measure, e.g. the maximum time delay error ratio.

A. Proof of margins in Sec. 2.2.1

Consider the robust lower bound and simple choice $T_c = \tau$ which gives $K_p = \frac{T}{2K\tau}$, and the case with $T_i = \min(T, 8\tau) = T$. The loop transfer function is

$$\begin{aligned} h_0(s) &= h_c(s)h_p(s) \\ &= K_p\frac{1+T_is}{T_is}K\frac{e^{-\tau s}}{1+Ts} = \frac{1}{2\tau}\frac{e^{-\tau s}}{s}. \end{aligned} \tag{130}$$

The frequency response is obtained by putting $s = j\omega$ where $\omega \geq 0$ is the frequency, i.e.,

$$\begin{aligned} h_0(j\omega) &= h_c(j\omega)h_p(j\omega) = \frac{1}{2\tau}\frac{e^{-j\tau\omega}}{j\omega} \\ &= |h_0(j\omega)|e^{j\angle h_0(j\omega)}, \end{aligned} \tag{131}$$

which is on polar form with magnitude, $|h_0(j\omega)|$, and phase angle, $\angle h_0(j\omega)$, given by

$$|h_0(j\omega)| = \frac{1}{2\tau\omega},\ \angle h_0(j\omega) = -\tau\omega - \frac{\pi}{2}. \tag{132}$$

This gives the phase crossover frequency (from $\angle h_0(j\omega_{180}) = -\pi$), i.e.

$$\omega_{180} = \frac{\pi}{2\tau}, \tag{133}$$

which gives the Gain Margin (GM) as in Sec. (2.2.1).

The SIMC PI controller tuning gives a gain crossover frequency, $\omega_c = \frac{1}{2\tau}$, so that $|h_p(j\omega_c)| = 1$. This results in a Phase Margin,

$$PM = \angle h_p(j\omega_c) + \pi = \frac{\pi - 1}{2} \approx 61.4°, \tag{134}$$

which gives the maximum time delay error as in eq. (7).

B. Proof of eq. (91)

The gain crossover frequency, ω_c, satisfies $|h_0(j\omega_c)| = 1$. Using eq. (88) for the magnitude we obtain

$$\frac{K_pk}{T_i\omega_c^2}\sqrt{1+(T_i\omega_c)^2} = 1. \tag{135}$$

This may be expressed as a 2nd order polynomial in ω_c^2, i.e.,

$$\left(\frac{T_i}{K_pk}\right)^2\omega_c^4 - T_i^2\omega_c^2 - 1 = 0. \tag{136}$$

Solving for ω_c^2 and using the positive solution we find

$$\omega_c^2 = \frac{1+\sqrt{1+\frac{4}{(K_pT_ik)^2}}}{2}(K_pk)^2. \tag{137}$$

And from this we find eq. (91) for the Gain crossover frequency.

C. Algorithm 6.1 MATLAB m-file

```
function [alfa,beta,PM,a,f]=pi_tun_maxdelay(c,delta)
% [alfa,beta,PM,a,f]=pi_tun_maxdelay(c,delta)
% On Input
% c=alfa*beta; - Method dependent product.
% delta - The prescribed maximum time delay error.
% On output
% alfa - Kp=alfa/(k*tau)
% beta - Ti=beta*tau
% PM   - The phase margin

f=(1+sqrt(1+4/(c)^2))/2;
a=atan(sqrt(f)*c)/sqrt(f);
beta=(c/a)*(delta+1);
alfa=a/(delta+1);
PM=delta*sqrt(f)*alfa;
% End pi_tun_maxdelay
```

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