Exercise to MATLAB Course lecture 3

David Di Ruscio

Telemark Institute of technology
N-3914 Porsgrunn, Norway
Fax: +47 35 57 52 50
Tel: +47 35 57 51 68
E-mail: David.Di.Ruscio@hit.no

1 Problem formulation

A chemical isothermal reactor (Van de Vusse) is studied in this example. The relationship from the feed flow rate u into the reactor to the concentration of the product y at the outlet of the reactor is modeled by the following non-linear state space model.

$$\dot{x_1} = -k_1 x_1 - k_3 x_1^2 + (v - x_1)u, \tag{1}$$
$$\dot{x_2} = k_1 x_1 - k_2 x_2 - x_2 u, \tag{2}$$
$$y = x_2, \tag{3}$$

where the reaction rate coefficients are given by $k_1 = 50$, $k_2 = 100$, $k_3 = 10$. The concentration of the by-product into the reactor, v, is treated as an unknown constant or slowly varying disturbance with nominal value $v^s = 10$.

The following steady state nominal (equilibrium) values are given:

$$v^s = 10, \tag{4}$$
$$u^s = 25, \tag{5}$$
$$x_1^s = 2.5, \tag{6}$$
$$y^s = x_2^s = 1. \tag{7}$$

The above model may be written as a vector Ordinary Differential Equation (ODE) model of the form

$$\dot{x} = f(x, u, v), \tag{8}$$
$$y = g(x), \tag{9}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x, u, v) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{10}$$

where

$$f_1 = -k_1 x_1 - k_3 x_1^2 + (v - x_1)u, \quad (11)$$

$$f_2 = k_1 x_1 - k_2 x_2 - x_2 u. \quad (12)$$

Furthermore the output equation is linear in this example due to $y = x_2$ in the model.

2 Exercise Tasks

Use MATLAB in all tasks below. Write an m-file script for the solutions. Write with advantage separate m-file scripts for Tasks 3 and 4 below.

1. Show that the given steady state nominal values, v^s, u^s, x_1^s, x_2^s, actually is equilibrium values so that $\dot{x}_1^s = 0$ and $\dot{x}_2^s = 0$.

2. We want to define the time interval

$$0 \leq t \leq t_f \quad (13)$$

   and with a step length parameter, $h = 0.001$, and final time instant $t_f = 0.3$. Define a vector

$$t = \begin{bmatrix} 0 & h & 2h & \cdots & t_f \end{bmatrix} \quad (14)$$

   in MATLAB using the : (column) operator. Find the number of discrete time instants, N, in the discrete time vector, t.

3. Simulate (solve) the differential equation $\dot{x} = f(x, u)$, $y = g(x)$ by using the explicit Euler method for integration. Implement the simulation in a for-loop. Store the solutions in data matrices X for the states x^T, Y for the output $y = y(t)$. Plot the results. Assume that the initial values for the states are $x_1(t = 0) = x_1^s$ and $x_2(t = 0) = x_2^s$. Specify a step in the input feed flow rate from $u = 25$ to $u = 30$ at time $t = \frac{t_f}{2}$.

4. Simulate (solve) the differential equation $\dot{x} = f(x, u)$, $y = g(x)$ by using the built in ODE solver, **ode15s** for the integration. Plot the results and compare with the Euler simulation results from the previous task. Assume that the initial values for the states are $x_1(t = 0) = 0$ and $x_2(t = 0) = 0$. Specify a step in the input feed flow rate from $u = 25$ to $u = 30$ at time $t = \frac{t_f}{2}$.

5. In the response from the input u to the output $y = x_2$ there is a so called "inverse response". Use the zoom MATLAB figure function to display this better. Plot the response from u to $y = x_2$ in one figure.

6. Assume that the model, $\dot{x} = f(x, u, v)$ is linearized around the nominal values x^s, u^s and v^s. Find the time constants of the system (around nominal values).

   Find the Jacobian system matrix

$$A_c = \frac{\partial f(x, u, v)}{\partial x^T}|_0 \tag{15}$$

   where $|_0$ means nominal values used in the arguments.

   - Find A_c and the time constants analytically. remember that the relationship between the eigenvalues, λ_1 and λ_2, and the time constants is, $T_1 = -\frac{1}{\lambda_1}$ and $T_2 = -\frac{1}{\lambda_2}$
   - Find matrix A_c numerically by using the function jacobi.m.

Solutions

Some background and solution of topics regarding this exercise is presented in the following.

A linearized model around steady state is given by

$$\Delta\dot{x} = A_c \Delta x + B_c \Delta u, \tag{16}$$

where $\Delta x = x - x^s$ and $\Delta u = u - u^s$ and

$$A_c = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x^s,u^s} = \begin{bmatrix} -k_1 - 2k_3 x_1^s - u^s & 0 \\ k_1 & -k_2 - u \end{bmatrix} = \begin{bmatrix} -125 & 0 \\ 50 & -125 \end{bmatrix}, \tag{17}$$

$$B_c = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{x^s,u^s} = \begin{bmatrix} v^s - x_1^s \\ -x_2^s \end{bmatrix} = \begin{bmatrix} 7.5 \\ -1 \end{bmatrix}. \tag{18}$$

A discrete time model is obtained by using a zero order hold on the input and a sampling interval $h = 0.002$, i.e.,

$$x_{k+1} = Ax_k + Bu_k + v, \tag{19}$$

$$y_k = Dx_k, \tag{20}$$

where

$$A = e^{Ach} = \begin{bmatrix} 0.7788 & 0 \\ 0.0779 & 0.7788 \end{bmatrix}, \tag{21}$$

$$B = A_c^{-1}(e^{A_c h} - I)B_c = \begin{bmatrix} 0.0133 \\ -0.0011 \end{bmatrix}, \tag{22}$$

$$D = \begin{bmatrix} 0 & 1 \end{bmatrix}, v = x^s - Ax^s - Bu^s + C(v - v^s). \tag{23}$$

Choosing an LQ criterion

$$J_i = \frac{1}{2}\sum_{k=i}^{\infty}(Q(y_k - r)^2 + P\Delta u_k^2), \tag{24}$$

with

$$P = 1, Q = 500, \tag{25}$$

gives the LQ-optimal control

$$u_k = u_{k-1} + G_1\Delta x_k + G_2(y_{k-1} - r), \tag{26}$$

where

$$G_1 = \begin{bmatrix} -23.4261 & -84.5791 \end{bmatrix}, G2 = -20.0581. \tag{27}$$

Simulation results after changes in the reference signal r are illustrated in Figure 1.

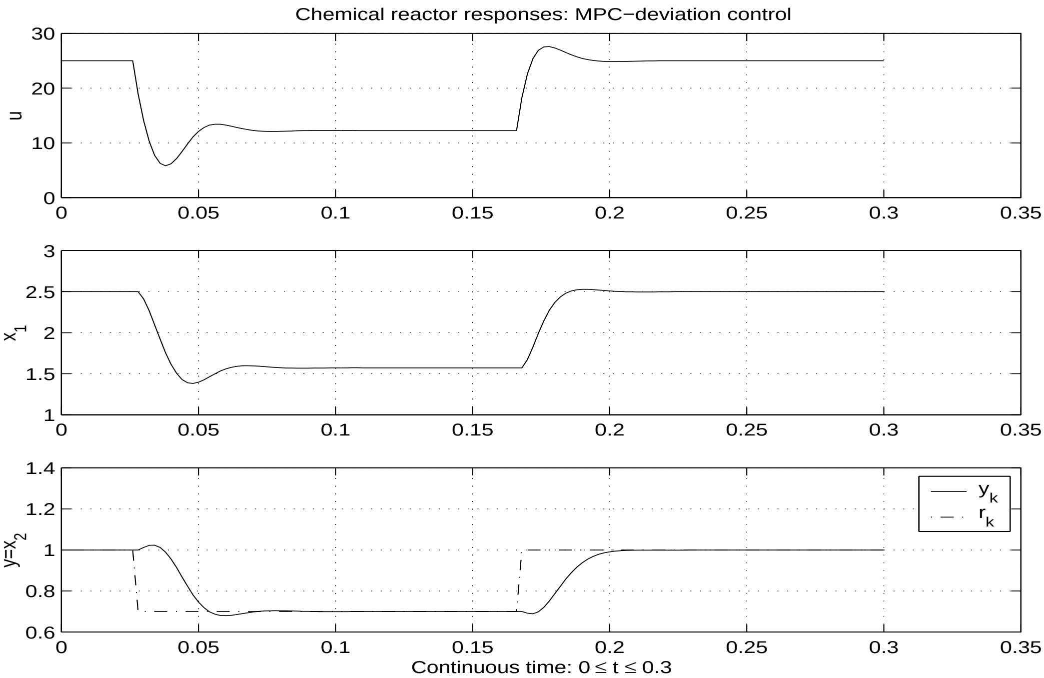

Figure 1: Simulation of the chemical reactor in Example **??** with LQ-optimal control. This figure is generated by the MATLAB file **dlq_ex3_du.m**.