Final Exam Course SCE4106 Model Predictive Control with Implementation Friday December 10, 2010 kl. 9.00 - 12.00

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Task 1 (30%) (MPC with Integral Action)

Consider a system described by the discrete state space model

$$x_{k+1} = Ax_k + Bu_k + v, \quad (1)$$
$$y_k = Dx_k + w, \quad (2)$$

where v and w are possibly unknown constant disturbances.
We want to design an MPC controller which minimizes the following criterion

$$J_k = \sum_{i=1}^{L}((y_{k+i} - r_{k+i})^T Q_i (y_{k+i} - r_{k+i}) + \Delta u_{k+i-1}^T P_i \Delta u_{k+i-1}), \quad (3)$$

where $\Delta u_k = u_k - u_{k-1}$ and r_k is a specified reference vector. $Q_i \geq 0$ and $P_i \geq 0$ are weighting matrices.

a) The above MPC criterion, (3), may be written in more compact form as

$$J_i = (y_{k+1|L} - r_{k+1|L})^T Q (y_{k+1|L} - r_{k+1|L}) + \Delta u_{k|L}^T P \Delta u_{k|L}. \quad (4)$$

Specify the matrices Q and P, and the vectors $y_{k+1|L}$, $r_{k+1|L}$ and $\Delta u_{k|L}$ in the objective (4).

b)

1. Show that it is possible to write the model in (1) and (2) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \quad (5)$$
$$\Delta y_k = D\Delta x_k, \quad (6)$$

where

$$\Delta x_k = x_k - x_{k-1}, \ \ \Delta u_k = u_k - u_{k-1}, \ \ \Delta y_k = y_k - y_{k-1}. \quad (7)$$

What can be gained by doing this?

2. Show that the model in (5) and (6) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \quad (8)$$
$$y_k = \tilde{D}\tilde{x}_k, \quad (9)$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix}. \quad (10)$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) From the augmented state space model (8) and (9) we define a Prediction Model (PM) of the form

$$y_{k+1|L} = p_L + F_L \Delta u_{k|L}. \tag{11}$$

Specify the matrix F_L and the vector p_L !

d) Show that the performance index (4) can be written as

$$J_k = \Delta u_{k|L}^T H \Delta u_{k|L} + 2 f_k^T \Delta u_{k|L} + J_0. \tag{12}$$

Specify the matrix H and the vector f_k !

e) Find the optimal unconstrained control deviation

$$\Delta u_{k|L}^* = f(\cdot), \tag{13}$$

and specify the function $f(\cdot)$! You should also specify the optimal MPC control, u_k^*, at present time k.

f) Assume that we have the following constraints

$$\begin{aligned} \Delta u_{k|L}^{min} \leq \Delta u_{k|L} &\leq \Delta u_{k|L}^{max}, && (14) \\ u_{k|L}^{min} \leq u_{k|L} &\leq u_{k|L}^{max}. && (15) \\ & && (16) \end{aligned}$$

Show that the constraints can be written as a linear inequality

$$\mathcal{A} \Delta u_{k|L} \leq b. \tag{17}$$

Specify the matrix $\mathcal{A}$ and the vector b in this case.

g) How can we solve the constrained MPC problem ?

Task 2 (25%) (Computing present state, x_k, and state observer)

Given a discrete time state space model

$$\begin{aligned} x_{k+1} &= A x_k + B u_k, && (18) \\ y_k &= D x_k. && (19) \end{aligned}$$

From the model (18) and (19) we may construct a Prediction Model (PM) which is a function of the present state, x_k. In case when the state vector x_k is not measured, we may use an estimate $\hat{x}_k$.

a) From the model (18) and (19) we construct a PM

$$y_{k+1|L} = p_L + F_L u_{k|L}. \tag{20}$$

Define expressions for the terms p_L and F_L in Eq. (20).

b) Show that the present state can be expressed by

$$x_k = A^{J-1} x_{k-J+1} + C^d_{J-1} u_{k-J+1|J-1}, \tag{21}$$

where J is a horizon into the past. Give an expression for the matrix C^d_{J-1} and the vector $u_{k-J+1|J-1}$. Tips: this equation may be deduced from (18).

In particular define the terms in Eq. (21) for $J = 4$.

c) Consider the matrix equation

$$y_{k|J} = O_J x_k + H^d_J u_{k|J-1}. \tag{22}$$

Define the matrices O_J and H^d_J. Use $J = 4$ as an example.

d) Use equations (21) and (22) in order to find an estimate of the present state of the form

$$\hat{x}_k = K_y \tilde{y} + K_u \tilde{u}. \tag{23}$$

Define the gain matrices K_y and K_u and the vectors $\tilde{y}$ and $\tilde{u}$!

e) Formulate an alternative state observer for constructing an estimate, $\hat{x}_k$, of the present state, x_k.

Task 3 (15%) Diverse MPC related questions

a) A Finite Impulse Response (FIR) model

$$y_k = h_1 u_{k-1} + h_2 u_{k-2} + h_3 u_{k-3}, \tag{24}$$

can be written as a state space model

$$x_{k+1} = A x_k + B u_k, \tag{25}$$

$$y_k = D x_k. \tag{26}$$

Define the model matrices A, B and D.

b) A step response model

$$y_{k+1} = a_0 y_k + h_1 \Delta u_{k-1} + h_2 \Delta u_{k-2} + h_3 \Delta u_{k-3}, \tag{27}$$

where $\Delta u_k = u_k - u_{k-1}$, can be written as a state space model

$$\begin{aligned} x_{k+1} &= A x_k + B \Delta u_k, & (28) \\ y_k &= D x_k. & (29) \end{aligned}$$

Define the model matrices A, B and D.

c) Given the polynomial model,

$$y_k = a_1 y_{k-1} + b_0 u_{k-1} + b_1 u_{k-2}. \tag{30}$$

Find a state space model equivalent

$$\begin{aligned} x_{k+1} &= A x_k + B u_k, & (31) \\ y_k &= D x_k. & (32) \end{aligned}$$

Define the model matrices A, B and D.