Discrete time Linear Quadratic Optimal Control

Dr. David Di Ruscio
Telemark Institute of technology
N-3914 Porsgrunn, Norway
E-mail: David.Di.Ruscio@hit.no

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1.2 The discrete maximum principle

Given a discrete time dynamic process described by the model

$$x_{k+1} - x_k = f(x_k, u_k, k), \tag{1.1}$$

where k is discrete time. $f(\cdot)$ is in general a nonlinear vector function.
Furthermore, we assume an optimal performance index (criterion) of the form

$$J_i = S(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k), \tag{1.2}$$

where $S(\cdot)$ is a scalar weighting function of the state at the final time instant N, $L(\cdot,\cdot)$ is a scalar weighting function of the state vector x_k and the control input vector u_k over the time horizon $i \le k \le N-1$. Both $S(\cdot)$ and $L(\cdot,\cdot)$ may be non linear functions.
By investigating this criterion we se that the discrete start time is $k = i$ and that the discrete final time is $k = N$. We assume that $N > i$. The criterion is defined over a time horizon of $N - i + 1$ discrete time instants. We also observe that the criterion only is dependent of the control inputs at $N - i$ time instants. Hence, this means that a part ov the criterion is not dependent of the unknown control inputs, and the criterion may be splitted into two parts. More of this later on.
We will in the following present the discrete time Maximum Principle which is a method for solving the discrete time optimal control problem
We define the discrete time Hamiltonian function corresponding to the continuous case. We have

$$\begin{aligned} H_k &= L(x_k, u_k) + p_{k+1}^T f(x_k, u_k, k) \\ &= L(x_k, u_k) + p_{k+1}^T (x_{k+1} - x_k). \end{aligned} \tag{1.3}$$

In order for the existence of an optimal control which minimize the criterion J_i it is necessary that:

- The impulse vector, p, and the state vector, x, satisfy the differential equations

$$x_{k+1} - x_k = \frac{\partial H_k}{\partial p_{k+1}} = f(x_k, u_k, k), \tag{1.4}$$

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k}, \tag{1.5}$$

  with known boundary (initial and final value) conditions

$$x_i = x_0, \tag{1.6}$$

$$p_N = \frac{\partial S}{\partial x_N}. \tag{1.7}$$

The state space model (1.1) have boundary conditions at the initial time instant. But remark that the model for the impulse vector (1.7) have boundary condition at the final time instant. This is defined as a two-point boundary value problem.

- The Hamiltonian function, H_k, must have an a absolute minimum (ore maximum) with respect to the unknown control $u_k \in U$ where U is the allowed control space. This must hold for all time instants $k = i, \cdots, N-1$. This means that we may include constraints on the control vector u_k. Those constraints define the control space U.

  Conditions for a minimum is that

$$\frac{\partial H_k}{\partial u_k} = 0, \tag{1.8}$$

  and

$$\frac{\partial^2 H_k}{\partial u_k^2} > 0. \tag{1.9}$$

1.3 Discrete optimal control of linear dynamic systems

Assume that the process may be described by the discrete time state space model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{1.10}$$

where $x_k \in \mathbb{R}^n$ is the state vector of the dynamic process and $u_k \in \mathbb{R}^r$ is the control vector. $A_k \in \mathbb{R}^{n \times n}$ is the transition matrix which in general may be time variant $B_k \in \mathbb{R}^{n \times r}$ is the control input system matrix.
Consider an ptimal criterion of the Linear Quadratic (LQ) form

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T P_k u_k), \tag{1.11}$$

where S_N, Q_k and P_k are symmetric weighting matrices. Note that the weighting matrices in general may be time variant. We will later on specify further detectability assumptions on the weighting matrices.
We will in the following find the optimal control, u_k^*, which minimize the optimal criterion Equation (1.11). We start by writing down the Hamiltonian function, i.e.,

$$H_k = \frac{1}{2}(x_k^T Q_k x_k + u_k^T P_k u_k) + p_{k+1}^T((A_k - I)x_k + B_k u_k). \tag{1.12}$$

We have used that the state space model equation (1.10) may be written as

$$x_{k+1} - x_k = (A_k - I)x_k + B_k u_k. \tag{1.13}$$

The optimal control is then given by

$$\frac{\partial H_k}{\partial u_k} = P_k u_k + B_k^T p_{k+1} = 0, \tag{1.14}$$

which may give

$$u_k = -P_k^{-1} B_k^T p_{k+1}. \tag{1.15}$$

if the weighting matrix is non-singular (invertible). One should note that we later on will present a version which does not involve the inversion of P_k.
Putting this into the state space model gives

$$x_{k+1} = A_k x_k - B_k P_k^{-1} B_k^T p_{k+1}. \tag{1.16}$$

We will later on use this expression for x_{k+1} in order for defining an expression for the optimal control. The impulse vector is defined from Equation (1.5). We have

$$p_{k+1} - p_k = -\frac{\partial H_k}{\partial x_k} = -Q_k x_k - (A_k - I)^T p_{k+1}, \tag{1.17}$$

which may be presented simply as

$$p_k = Q_k x_k + A_k^T p_{k+1}. \tag{1.18}$$

Equations (1.16) and (1.18) defines an autonomous system, i.e.,

$$\begin{bmatrix} x_{k+1} \\ p_k \end{bmatrix} = \begin{bmatrix} A_k & -H \\ Q_k & A_k^T \end{bmatrix} \begin{bmatrix} x_k \\ p_{k+1} \end{bmatrix}, \tag{1.19}$$

where the matrix H is defined as

$$H = B_k P_k^{-1} B_k^T. \tag{1.20}$$

This matrix should not be compared with the Hamiltonian function H_k.
Note that in Equation (1.19) the state vector and the impulse vector are defined at different time instants at the same side of the equality sign. In case when A_k is non-singular we find from (1.16) that

$$x_k = A_k^{-1} x_{k+1} + A_k^{-1} H p_{k+1}. \tag{1.21}$$

Putting this into (1.18) we find that

$$p_k = Q_k A_k^{-1} x_{k+1} + (A_k^T + Q_k A_k^{-1} H) p_{k+1}. \tag{1.22}$$

Equationse (1.21) and (1.22) may be written in matrix form as follows

$$\begin{bmatrix} x_k \\ p_k \end{bmatrix} = \overbrace{\begin{bmatrix} A_k^{-1} & A_k^{-1} H \\ Q_k A_k^{-1} & A_k^T + Q A_k^{-1} H \end{bmatrix}}^{F} \begin{bmatrix} x_{k+1} \\ p_{k+1} \end{bmatrix}. \tag{1.23}$$

Note that the transition matrix A_k is invertible if the model is obtained by discretizing a continuous time model. You should note that (1.23) may be used in order to show that there is a linear relationship between pk and x_k, i.e., $p_k = R_k x_k$ as well as to find an equation for R_k.
The prof of this is as follows. From (1.7) we find the boundary condition $p_N = S_N x_N$. This indicates that there is a linear relationship between x_k and p_k. Putting $k = N-1$ in (1.23) gives, with using the boundary conditions, two equations with three unknown, p_{N-1}, x_{N-1} og x_N. Eliminating x_N we find the linear relationship

$$p_{N-1} = R_{N-1} x_{N-1}, \tag{1.24}$$
$$R_{N-1} = (F_{21} + F_{22} S_N)(F_{11} + F_{12} S_N)^{-1}. \tag{1.25}$$

Putting $k = N-2$ into (1.23) and doing the same, i.e., finding a linear relationship between p_{N-2} and x_{N-2}. Since that we have a series to do, we use the induction principle for the prof, i.e., we can prove that there is a linear relationship between p_k and x_k. We will later on generalize this to hold also when A_k is singular.
In the same way as in the continuous case, and which is sketched above, we may show that there is a linear relationship between the impulse vector, p_k, and the state vector, x_k. Hence, we may show and assume that

$$p_k = R_k x_k. \tag{1.26}$$

This means that if we may find an equation for defining/computing R_k then we indeed have proved that there exist such a relationship as described above. This also indicates an alternative prof of the LQ optimal solution to the one given above. This prof is presented in the following
Putting (1.18) into (1.26) gives

$$R_k x_k = Q_k x_k + A_k^T p_{k+1}. \tag{1.27}$$

Expressing (1.26) at time instant $k+1$ and putting this expression into (1.27) we find

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} x_{k+1}. \tag{1.28}$$

We will now find an expression for x_{k+1} and putting this into (1.28). Putting the relationship (1.26) into (1.16) gives

$$x_{k+1} = A x_k - B_k P_k^{-1} B_k^T R_{k+1} x_{k+1}. \tag{1.29}$$

From this last equation we find an expression for for x_{k+1}

$$x_{k+1} = (I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k x_k. \tag{1.30}$$

Note that (1.30) have to be an expression for the closed loop system. Putting equation (1.30) into (1.28) gives

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} (I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k x_k. \tag{1.31}$$

This equation must hold for an arbitrarily state vector $x_k \neq 0$. This gives the following matrix equation for finding R_k.

$$R_k = Q_k + A_k^T R_{k+1}(I + B_k P_k^{-1} B_k^T R_{k+1})^{-1} A_k. \quad (1.32)$$

This is one formulation of the famous Riccati equation named after Count Riccati which lived in the 1600 century. However, this formulation assumes that the control weighting matrix, P_k, is non-singular. We will later show that there exist a more general formulation of the discrete Riccati equation wich does not involve the inversion of P_k.
An alternative formulation in the case when R_{k+1} is non-singular is

$$R_k = Q_k + A_k^T (R_{k+1}^{-1} + B_k P_k^{-1} B_k^T)^{-1} A_k. \quad (1.33)$$

From (1.7) we find the boundary condition

$$p_N = S_N x_N. \quad (1.34)$$

Expressing the relationship (1.26) at $k = N$ we find that

$$p_N = R_N x_N. \quad (1.35)$$

Comparison of (1.34) and (1.35) gives the boundary condition

$$R_N = S_N, \quad (1.36)$$

which gives the boundary condition for the discrete time Riccati equation. This means that the solution R_k (at time k) may be found by iterating the Riccati equation backward in time, to the present time instant k, from the final time instant, $k = N$.
An expression for the optimal control can now be found by putting (1.26) into (1.15), i.e.,

$$u_k = -P^{-1} B^T R_{k+1} x_{k+1}. \quad (1.37)$$

Putting (1.30) into (1.37) gives

$$u_k = G_k x_k, \quad (1.38)$$

$$G_k = -P^{-1} B^T R_{k+1}(I + B P^{-1} B^T R_{k+1})^{-1} A. \quad (1.39)$$

As we see, the above solution assumes that the weighting matrix P_k is non-singular. We will in the next section propose a better solution which does not involve the inversion of P_k.
Consider now the case in which the time horizon is larghe, i.e., $N \to \infty$, then we have that $R_{k+1} = R_k = R$ is a constant matrix. This gives us the Discrete time Algebraic Riccati Equation (DARE). Furthermore, we may show that when chosing the weighting matrices properly then the LQ optimal solution results in a stable closed loop system. In general we have that the LQ optimal control system is stable when $N \to \infty$, under the assumptions that (A, B) is stabilizable, $(\sqrt{Q}, A)$ is detectable and P a positive definite matrix. As mentioned above, there may also in certain circumstances exist an LQ optimal solution also when P is singular.

1.3.1 Derivation of the optimal control: intuitive formulation

The solution to the discrete time LQ optimal control problem may be formulated in different ways and with different equations. In case when the transition matrix A_k is non-singular then we may find p_{k+1} from Equation (1.18), i.e.,

$$p_{k+1} = A^{-T}(p_k - Q_k x_k) = A^{-T}(R_k - Q_k)x_k, \tag{1.40}$$

where we have assumed that $p_k = R_k x_k$. Putting this into the expression for the optimal control given by Equation (1.15), we find

$$u_k = G_k x_k, \tag{1.41}$$
$$G_k = -P_k^{-1} B_k^T A_k^{-T}(R_k - Q_k). \tag{1.42}$$

This solution demands that both A_k and P_k are non-singular matrices. A_k is usually non-singular. This is in particular the case when A_k is found from discretizing a continuous time model. There may however exist cases in which A_k is singular. This is the case for systems with a static component and for systems with time delay modeled as extra "dummy" states in the system in order to take care of the time delay.

1.3.2 Derivation of the optimal control: a better formulation

We may show that there exist a formulation of the discrete LQ optimal solution which does not involve the inversion of the matrices A_k and P_k. We have from the condition for a minimum, equation (1.14), that

$$P_k u_k = -B_k^T R_{k+1} x_{k+1}, \tag{1.43}$$

where we have assumed $p_{k+1} = R_{k+1} x_{k+1}$. Putting the state space model into (1.43) gives

$$P_k u_k = -B_k^T R_{k+1}(A_k x_k + B_k u_k). \tag{1.44}$$

This gives

$$(P_k + B_k^T R_{k+1} B_k) u_k = -B_k^T R_{k+1} A_k x_k. \tag{1.45}$$

This gives the following nice expression for the optimal control

$$u_k^* = G_k x_k, \tag{1.46}$$
$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k. \tag{1.47}$$

R_{k+1} may be found from the Riccati equation (1.32) or (1.33). However, we will in the next section derive a 3rd formulation of the discrete time Riccati equation which is to be preferred compared to Equations (1.32) and (1.33).

1.3.3 Alternative formulations of the discrete time Riccati equation

The discrete time Riccati equation in the LQ optimal control solution may be formulated in different ways. In Section (1.3) we have derived two different formulations. Se Equations (1.32) and (1.33). We will in this section propose two different formulations which does not involve the inversion of the weighting matrix P_k. These formulations are may be the most used formulations.
The starting point is as shown earlier, i.e., by putting Equation (1.18) into (1.26), we have

$$R_k x_k = Q_k x_k + A_k^T R_{k+1} x_{k+1}, \tag{1.48}$$

where we have used that at $p_{k+1} = R_{k+1} x_{k+1}$.
An expression for the closed loop system is obtained by putting the optimal control (1.46) and (1.47) into the discrete time state Equation $x_{k+1} = A_k x_k + B_k u_k$. This gives

$$x_{k+1} = (A_k - B_k(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k) x_k. \tag{1.49}$$

Putting (1.49) into (1.48) gives

$$R_k x_k = Q_k x_k + A_k^T R_{k+1}(A_k - B_k(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k) x_k. \tag{1.50}$$

This equation must hold for all states $x_k \neq 0$. Hence we have,

$$R_k = Q_k + A_k^T (R_{k+1} - R_{k+1} B_k (P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}) A_k. \tag{1.51}$$

This formulation of the discrete time Riccati equation is to be preferred. As we see, only the matrix $P_k + B_k^T R_{k+1} B_k$ have to be inverted. Note that the boundary condition is as before, i.e. $R_N = S_N$.
Finally, we will present a 4th formulation of the Riccati equation. Hence, we may show that

$$R_k = (A_k + B_k G_k)^T R_{k+1} (A_k + B_k G_k) + G_k^T P_k G_k + Q_k, \tag{1.52}$$
$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k. \tag{1.53}$$

This formulation of the discrete time Riccati equation is known in the litterature as the Josephs stable version of the Riccati equation. As we see, this Riccati equation consists only of symmetric terms. This formulation is to be preferred in numerical calculations.
We also se that for a given control gain matrix, G_k, then Equation (1.52) is a discrete time Lyapunov equation. Equations (1.52) and (1.53) can with advantage be used in order to iterate to find the stationary solution to the LQ optimal control problem, i.e. the problem with infinite horizon $N \to \infty$.
Note that the boundary conditions to the different formulations of the Riccati equation is the same, i.e., $R_N = S_N$ where S_N is the weighting matrix for the final state, x_N.

1.3.4 Numerical example

Example 1.1 (Singular transition matrix)
Given a system described by a linear discrete state space model with the following model matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \; D = \begin{bmatrix} 1 & -1 \end{bmatrix}, \tag{1.54}$$

and with weighting matrices

$$P = 1, \; Q = D^T D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; S_N = Q. \tag{1.55}$$

We chose the following initial value for the state vector, i.e.,

$$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{1.56}$$

and simulate the optimal closed loop system over the time horizon $i \leq k \leq N$ where $i = 0$ and $N = 5$. This gives after $N = 5$ iterations of the Riccati equation (1.53)

$$R_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1.4993 \end{bmatrix}, R_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1.497 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1.488 \end{bmatrix}, \tag{1.57}$$

$$R_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1.455 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1.333 \end{bmatrix}, R_5 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.58}$$

and where $R_5 = S_5$ is defined from the specified final boundary value condition. It can be shown, se Pappas og Laub (1980), that the solution of the stationary discrete Riccati equation, i.e. the solution when $N \to \infty$, is given by

$$R = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}. \tag{1.59}$$

In general we have that $\lim_{N \to \infty} R_0 = R$. We se that even for a "short" horizon as $N = 5$ then R_0 is a relatively good approximation to the stationary solution, for this example.
Furthermore, the optimal time variant feedback matrices are given by

$$G_k = \begin{bmatrix} 0 & \frac{\sqrt{2}}{1+2r_{22,k+1}} \end{bmatrix} \; \forall \; k = 0, \ldots, 4 \tag{1.60}$$

where $r_{22,k+1}$ is the lower right element in R_{k+1}. This means that the optimal control is given by a feedback

$$u_k = \frac{\sqrt{2}}{1 + 2r_{22,k+1}} x_{2,k} \tag{1.61}$$

where $x_{2,k}$ is the 2nd state in the state vector (1.56). For this system it is optimal to only take feedback from one of the two states in the system. This is

unusual because it in general is optimal with a feedback from all states in the system.
We remark that the system (A, B) is controllable and that (D, A) is observable. One special remark is that the system have two poles (eigenvalues) in origo. This means that the open loop system has infinite fast dynamics. The optimal system minimizes the objective J_i. The objective will in general obtain a small value if the state x_k goes fast to zero. It is therefore not optimal to make the system slower then necessary.
Simulations of the optimal control $u_k = G_k x_k$ and x_k is shown in Figure 1.1.
We end this example by mentioning that for systems with transport delay modeled as extra states, then the transition matrix will have eigenvalues in origo, and the optimal control will have a structure relatively equal to the above example.
△

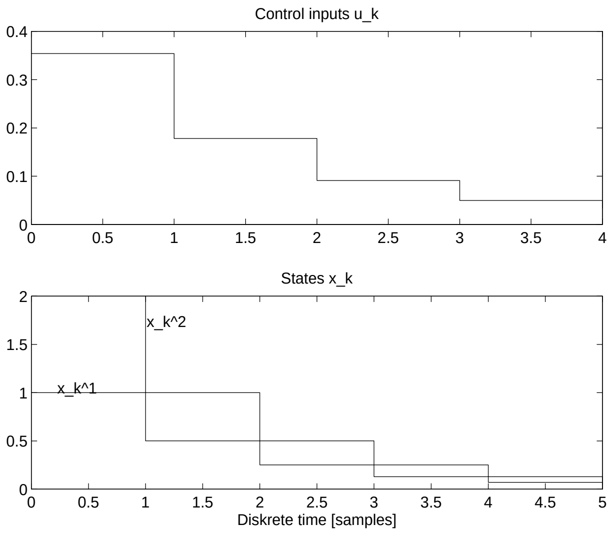

Figure 1.1: The Figure illustrates simulations of u_k and x_k for example 1.1. The discrete initial time is $i = 0$ and the final time instant is $N = 5$.

1.3.5 Summing up

We will summing up the results in this section in the following theorem

Theorem 1.3.1 (Discrete time Linear Quadratic optimal regulator) Given the discrete time system

$$x_{k+1} = A_k x_k + B_k u_k, \tag{1.62}$$

where $k \geq i$ and the initial value of the state vector, x_i, is given.
Consider given a LQ criterion valid over the time horizon $i \leq k \leq N$, i.e.,

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T P_k u_k), \tag{1.63}$$

where S_N, Q_k and P_k are symmetric weighting matrices.
The optimal control vector, u_k^*, which is minimizing the LQ criterion, J_i, is given by

$$u_k = G_k x_k, \tag{1.64}$$

$$G_k = -(P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1} A_k, \tag{1.65}$$

where R_{k+1} is the positive solution to the discrete time Riccati equation

$$R_k = Q_k + A_k^T (R_{k+1} - R_{k+1} B_k (P_k + B_k^T R_{k+1} B_k)^{-1} B_k^T R_{k+1}) A_k, \tag{1.66}$$

with final value boundary condition

$$R_N = S_N. \tag{1.67}$$

Furthermore, the minimum value of the criterion, J_i, is given by

$$J_i = \frac{1}{2} x_i^T R_i x_i. \tag{1.68}$$

and where R_i is found from the Riccati equation. $\triangle$

Merknad 1.1 *In some references it is common to define the state feedback matrix as $K_k = -G_k$, and $u_k = -K_k x_k$ instead of $u_k = G_k x_k$ as in these lecture notes. This is in particular the case as e.g. in Lewis and Syrmos (1995). The MATLAB Control System Toolbox also uses the notation $K = -G$, se e.g. the* **dlqr** *function.*