On the sensitivity index

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Abstract

A discussion of some common robustness measures are given. The sensitivity index M_s is presented. An interpretation of the Gain Margin (GM) and Phase Margin (PM) are given.

1 The sensitivity index M_s

Define the sensitivity transfer function

$$S(s) = \frac{1}{1 + h_0(s)}, \tag{1}$$

where $h_0(s)$ is the loop transfer function. From this we find the frequency response of the sensitivity transfer function as

$$S(j\omega) = \frac{1}{1 + h_0(j\omega)}, \tag{2}$$

where $h_0(j\omega)$ is the frequency response of the loop transfer function.

Definition 1.1 (Sensitivity index M_s)
The sensitivity index M_s is defined as the maximum peak of the magnitude of the frequency response of the sensitivity transfer function, i.e.,

$$M_s = \max_{0 \leq \omega \leq \infty} |S(j\omega)| \tag{3}$$

Reasonable values of M_s is in the range $1.3 \leq M_s \leq 2.0$ where $M_s = 1.3$ is for a more robust and conservative controller setting than $M_s = 2.0$. Furthermore, in order to ensure stability we must have $M_s < \infty$.

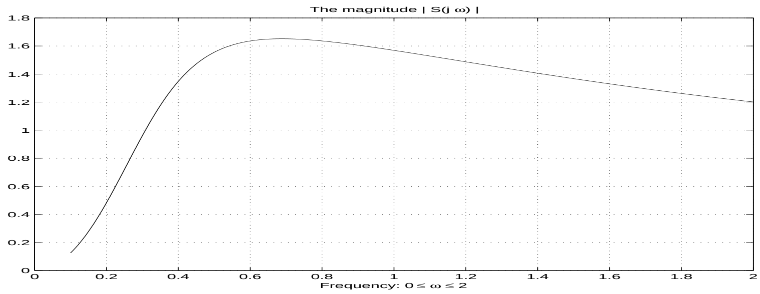

Figure 1: PI control of integrating plus time delay process, $h_p(s) = k\frac{e^{-\tau s}}{s}$ with gain $k = 1$ and time delay $\tau = 1$. PI-controller $h_c(s) = K_p\frac{1+T_i s}{T_i s}$ with settings $K_p = 0.4004$ and $T_i = 4.9953$. The figure illustrates the magnitude of the frequency response of the sensitivity transfer function as a function of frequency. The sensitivity index M_s, i.e., the maximum peak and for this system given by $M_s \approx 1.6515$.

The quantity $\frac{1}{M_s}$ is the shortest distance from the Nyquist curve to the critical point -1 in the complex plane. The Nyquist curve is obtained by plotting the frequency response of the loop transfer function in the complex plane. The complex number $h_0(j\omega) = a + jb = |h_0(j\omega)|e^{j\angle(h_0(j\omega)}$ may be plotted in the complex plane as a function of the frequency $0 \leq \omega \leq \infty$. When the frequency ω goes from 0 to ∞, the endpoint of the vector describes a curve in the complex plane which is the Nyquist curve.

An analytic expression for the sensitivity index seems difficult. However, an expression for the magnitude of the sensitivity function was presented in Di Ruscio (1992) as

$$|S(j\omega)| = \frac{1}{\sqrt{1 + |h_0(j\omega)|^2 + 2|h_0(j\omega)|\cos(\angle(h_0(j\omega))}} \tag{4}$$

2 Interpretation of the Gain Margin (GM)

We will in this section show that the Gain Margin (GM) is related to a multiplicative uncertainty in the process gain.

The Gain Margin (GM) is defined as

$$GM = \frac{1}{|h_0(j\omega_{180})|} \tag{5}$$

where $h_0(j\omega) = |h_0(j\omega)|e^{j\angle h_0(j\omega)}$ is the loop transfer function, $|h_0(j\omega)|$ is the magnitude and ω_{180} is the phase crossover frequency such that the phase angle is $\angle h_0(j\omega_{180}) = -\pi$.

The frequency response of the loop transfer function, $h_0(j\omega)$, is a function of the model and we may split out the process gain, k, as follows

$$h_0(j\omega) = kh(j\omega). \tag{6}$$

A model is never equal to the true process and assume now that the true process have an uncertainty in the process gain such that

$$h_0^p(j\omega) = k^p h(j\omega), \tag{7}$$

is the frequency response of the true loop transfer function and where, k^p, is the true process gain.

From the Bode stability criterion ($|h_0(j\omega_{180})| < 1$ for stability) we have that the feedback system is stable if the Gain Margin of the true process satisfy

$$GM^p = \frac{1}{|h_0^p(j\omega_{180})|} = \frac{1}{|k^p h(j\omega_{180})|} = \frac{k}{|k^p h_0(j\omega_{180})|} = \frac{kGM}{k^p} > 1 \tag{8}$$

We have in Eq. (8) used Eqs. (5) - (7).

From Eq. (8) we have that the true feedback system is stable if the true process gain, k^p, is bounded from up by

$$k^p < kGM \tag{9}$$

3 Interpretation of the Phase Margin (PM)

The SIMC PI controller tuning yields a Phase Margin, $PM \approx 61.4°$, for a first order system $h_p(s) = K\frac{e^{-\tau s}}{1+Ts}$, with the SIMC PI controller settings $T_i = \min(T, 2(T_c + \tau) = T$ and $K_p = \frac{T}{K(T_c+\tau)}$.

Furthermore we may tolerate a maximum time delay error,

$$d\tau_{\max} = \frac{PM}{\omega_c} = (\pi - 1)\tau = 2.14\tau. \tag{10}$$

One interpretation of this is as follows. Suppose that the true time delay, τ_p, in the process is, $\tau_p = \tau + d\tau$, where τ is the time delay in the model. The corresponding true Phase Margin is then $PM_p = -(\tau + d\tau)\omega_c - \frac{\pi}{2} + \pi = PM - d\tau\omega_c$. The maximum time delay error perturbation, $d\tau_{\max}$, which may be tolerated before the system becomes unstable is found for the phase margin limit ($PM_p = 0$), i.e., $PM_p = PM - d\tau_{\max}\omega_c = 0$, which gives eq. (10).