Sivilingeniørutdanningen ved
Institutt for prosessautomatisering
Avdeling for teknologiske fag
Høgskolene i Telemark
DDiR, August 31, 2006

SCE1106 Control Theory

Solution exercise 1

Solution exercise 1

Given

$$\dot{x} = Ax + Bu \tag{1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k_1 & k_2 \\ 0 & 0 & -k_3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ k_4 \end{bmatrix} \tag{2}$$

and $k_1 = 1$, $k_2 = 2$, $k_3 = 4$ og $k_4 = 2$

1 Eigenvalues

The system matrix A is upper triangular. The eigenvalues are then directly given by the diagonal elements of A. Hence, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -k_1$ and $\lambda_3 = -k_3$. An eigenvalue matrix is then given by

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -k_1 & 0 \\ 0 & 0 & -k_3 \end{bmatrix} \tag{3}$$

2 The eigenvectors

An eigenvector corresponding to $\lambda_1 = 0$

$$Am_1 = \lambda_1 m_1 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

This gives the equations

$$\begin{array}{rcrcl} m_{21} & & & = & 0 \\ -m_{21} & + & 2m_{31} & = & 0 \\ & & -4m_{31} & = & 0 \end{array} \tag{5}$$

This gives $m_{21} = 0$ and $m_{31} = 0$. m_{11} is arbitrarily and can be chosen freely in such a way that $m_{11} \neq 0$, e.g. choosing $m_{11} = 1$ gives

$$m_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Eigenvector corresponding to $\lambda_2 = -1$

$$Am_1 = \lambda_1 m_1 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = -1 \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} \tag{7}$$

Hence, we have the following equations

$$\begin{array}{rcrcr} m_{21} & & & = & -m_{11} \\ -m_{21} & + & 2m_{31} & = & -m_{21} \\ & & -4m_{31} & = & -m_{31} \end{array} \tag{8}$$

which can be written as

$$\begin{array}{rcrcr} m_{21} & + & m_{21} & = & 0 \\ & & 2m_{31} & = & 0 \\ & & -3m_{31} & = & 0 \end{array} \tag{9}$$

This gives $m_{31} = 0$ and $m_{11} = -m_{21}$. Choosing $m_{21} \neq 0$, e.g. $m_{21} = 1$ gives

$$m_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{10}$$

Eigenvector corresponding to $\lambda_3 = -4$

$$Am_1 = \lambda_1 m_1 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = -4 \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} \tag{11}$$

This gives the equations

$$\begin{array}{rcrcr} m_{21} & & & = & -4m_{11} \\ -m_{21} & + & 2m_{31} & = & -4m_{21} \\ & & -4m_{31} & = & -4m_{31} \end{array} \tag{12}$$

which can be written as

$$\begin{array}{rcrcrcr} 4m_{11} & + & m_{21} & & & = & 0 \\ & & 3m_{21} & + & 2m_{31} & = & 0 \\ & & & & 0 & = & 0 \end{array} \tag{13}$$

This gives $m_{21} = -4m_{11}$ and $m_{31} = 6m_{11}$. Choosing $m_{11} \neq 0$, e.g. $m_{11} = 1$ gives

$$m_3 = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix} \tag{14}$$

An eigenvector matrix, M, corresponding to the eigenvalue matrix, Λ, is then given by

$$M = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 6 \end{bmatrix} \tag{15}$$

3 Controlling the answer

The eigenvalue decomposition is such that

$$M^{-1}AM = \Lambda \tag{16}$$

This can be written as

$$AM = M\Lambda \tag{17}$$

Hence, the answer can be checked by computing the left and right hand sides AM og $M\Lambda$ and comparing the results.
If we chose to check if $M^{-1}AM = \Lambda$, then we need to compute the inverse M^{-1}, i.e.,

$$M^{-1} = \frac{1}{\det M}(\text{cof}M)^T \tag{18}$$

The cofactor matrix. cof$M$, is given by:

$$\text{cof}M = \begin{bmatrix} 6 & 0 & 0 \\ +6 & 6 & 0 \\ 3 & +4 & 1 \end{bmatrix} \tag{19}$$

where + indicates where sign are changed in the matrix of sub determinants.
The inverse of the eigenvector matrix is then given by

$$M^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ +6 & 6 & 0 \\ 3 & +4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \tag{20}$$

Solution exercise 2

The transfer function model is given by

$$y(s) = (D(sI - A)^{-1}B + E)u(s) \tag{21}$$

This system has two inputs u_1 and u_2 and one output y. The transfer function can then be written as

$$y(s) = h_1(s)u_1(s) + h_2(s)u_2(s) \tag{22}$$

where

$$h_1(s) = \frac{s+3}{s^2+3s+2} \tag{23}$$

$$h_2(s) = \frac{1}{s^2+3s+2} \tag{24}$$

this can be computed in MATLAB as follows:

$$[\text{teller}_1, \text{nevner}_1] = \text{ss2tf}(A, B, D, E, 1);$$
$$[\text{teller}_2, \text{nevner}_2] = \text{ss2tf}(A, B, D, E, 2);$$

where teller_1 and nevner_1 is the coefficients in the denominater and the numerator polynomials of $h_1(s)$, respectively. Similarly, teller_2 and nevner_2 are the coefficients of the denominator and the numerator polynomials in $h_2(s)$, respectively.

Solution exercise 3

See the similar example 2.1 in the lecture notes.