Task 1

a)

$$y_{k+1} - e_{k+1} = a(y_k - e_k) + bu_k + k e_k$$

$$\Downarrow$$

$$y_{k+1} = a y_k + b u_k + k e_k - a e_k + e_{k+1}$$

When $k = a$

$$y_k = a y_{k-1} + b u_{k-1} + e_k = \underbrace{\begin{bmatrix} y_{k-1} & u_{k-1} \end{bmatrix}}_{\varphi_k^T} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta} + e_k$$

Then

$$\underline{\underline{\varphi_k = \begin{bmatrix} y_{k-1} \\ u_{k-1} \end{bmatrix}}}, \quad \underline{\underline{\theta = \begin{bmatrix} a \\ b \end{bmatrix}}}$$

b) From Ex. 9.1 lectnotes PEM p 34.

We find $y_k = \varphi_k^T \theta + e_k$

whee

$$y_k = T \cdot \ln(p)$$

$$\underline{\underline{\varphi_k = \begin{bmatrix} -\ln(p) \\ T \\ 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} c \\ a \\ ac - b \end{bmatrix}}}$$

c) Divide up the sum

$$\bar{y}_t = \frac{1}{t}\left(\sum_{k=1}^{t-1} y_k + y_t\right)$$

Using

$$\bar{y}_{t-1} = \frac{1}{t-1}\sum_{k=1}^{t-1} y_k \Rightarrow \sum_{k=1}^{t-1} y_k = (t-1)\,\bar{y}_{t-1}$$

This gives

$$\bar{y}_t = \frac{1}{t}\Big((t-1)\,\bar{y}_{t-1} + y_t\Big) = \bar{y}_{t-1} + \frac{1}{t}\left(y_t - \bar{y}_{t-1}\right)$$

Hence

$$\underline{\underline{\bar{y}_t = \bar{y}_{t-1} + \frac{1}{t}\left(y_t - \bar{y}_{t-1}\right)}}$$

d) From known impulse responses

$$H_{212} = \begin{bmatrix} H_2 & H_3 & - & - & - & H_8 & H_9 \\ H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} \end{bmatrix} = O_2 A C_8$$

$$H_{112} = \begin{bmatrix} H_1 & \cdot & \cdot & \cdot & H_8 \\ H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \end{bmatrix} = O_2 C_8$$

$$O_2 = \begin{bmatrix} D \\ DA \end{bmatrix}, \quad C_8 = \begin{bmatrix} B & AB & A^2B & \cdots & A^7B \end{bmatrix}$$

$$n = \text{rank}(H_{112}) \quad H_{112} = U_1 S_1 V_1^T \quad \begin{cases} O_2 = U_1 \\ C_8 = S_1 V_1^T \end{cases}$$

Task 2

a)

$$y_k = -a_2 y_{k-1} - a_1 y_{k-2} + b_1 u_{k-1} + b_2 u_{k-2} + e_k$$

when

$$k_1 = -a_2 \quad , k_2 = -a_1$$

Linear regression model

$$y_k = \overbrace{\begin{bmatrix} -y_{k-1} & -y_{k-2} & u_{k-1} & u_{k-2} \end{bmatrix}}^{\varphi_k^T} \overbrace{\begin{bmatrix} a_2 \\ a_1 \\ b_1 \\ b_2 \end{bmatrix}}^{\theta} + e_k$$

b)

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \varphi_k \Lambda \varphi_k^T \right)^{-1} \sum_{k=1}^{N} \varphi_k \Lambda y_k$$

c)

$$K_t = P_t \varphi_t \Lambda$$

and

$$P_t^{-1} = P_{t-1}^{-1} + \varphi_t \Lambda \varphi_t^T$$

Task 3

a)

- $$\bar{y}_k = D\bar{x}_k + Eu_k \quad (1)$$
  $$\hat{x}_k = \bar{x}_k + K_k(y_k - \bar{y}_k) \quad (2)$$
  $$\bar{x}_{k+1} = A\hat{x}_k + Bu_k \quad (3)$$
  Apriori - aposteriori form.

- $$K_k = \bar{X}_k D^T (D\bar{X}_k D^T + W)^{-1}$$
  Kalman gain matrix.

- Putting (2) in (3), eliminating $\hat{x}_k$

  $$\underline{\underline{\bar{x}_{k+1} = A\bar{x}_k + Bu_k + \tilde{K}_k \varepsilon_k}}$$

  when $\tilde{K}_k = AK_k$ and $\varepsilon_k = y_k - \bar{y}_k$

  and then

  $$\underline{\underline{y_k = \underbrace{D\bar{x}_k + Eu_k}_{\bar{y}_k} + \varepsilon_k}}$$

  Innovations form of Kalman filter.

- $$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + \tilde{K}_k(y_k - \bar{y}_k)$$
  $$\bar{y}_k = D\bar{x}_k + Eu_k$$
  Kalman filter on prediction form