Final Exam Course SCE4106 Model Predictive Control with Implementation Friday 14th December 2007 kl. 9.00 - 12.00

Sluttprøven består av: 3 oppgaver.
Oppgaven teller 70 % av sluttkarakteren.
Det er 4 sider i sluttprøven.
Tillatte hjelpemidler: vedlegg til oppgaven

Faglig kontakt under eksamen:
Navn: David Di Ruscio
Tlf: 51 68, Rom: B249

Kybernetikk og industriell IT
Institutt for elektro, IT og kybernetikk
Avdeling for teknologiske fag
Høgskolen i Telemark
N-3914 Porsgrunn

Task 1 (25%): MPC

Given a process described by the linear discrete time state space model

$$x_{k+1} = Ax_k + Bu_k + Cr_k, \quad (1)$$

$$y_k = Dx_k, \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^r$ is the control vector and $y_k \in \mathbb{R}^m$ is the output vector. Here r_k is a reference vector.
Note: The model (1) and (2) may be the result of augmenting an integrator into a state space model.
Consider also the control objective

$$J_k = \sum_{i=1}^{L}((y_{k+i} - r_{k+i})^T Q_i (y_{k+i} - r_{k+i}) + (u_{k+i-1} - u_i)^T P_i (u_{k+i-1} - u_i)) \quad (3)$$

where $Q_i \in \mathbb{R}^{m \times m}$ and $P_i \in \mathbb{R}^{r \times r}$ are symmetric weighting matrices. The vector u_i represents desired values of the controls at time $k = i$.

a) Show that the control objective can be written as follows

$$J_k = (y_{k+1|L} - r_{k+1|L})^T Q (y_{k+1|L} - r_{k+1|L}) + (u_{k|L} - u_0)^T P (u_{k|L} - u_0). \quad (4)$$

Specify the extended vectors $r_{k+1|L}$, $y_{k+1|L}$ and $u_{k|L}$, as well as the extended weighting matrices Q and P, and the vector u_0.

b) Show that the process model (1) and (2) can be written as a prediction model of the form,

$$y_{k+1|L} = F_L u_{k|L} + p_L. \quad (5)$$

Specify expressions for F_L and p_L.

c)

- Find the MPC optimal unconstrained future controls of the form

$$u^*_{k|L} = f_1(\cdot). \quad (6)$$

  Specify the function $f_1(\cdot)$

- Which optimal MPC control, u^*_k, is used to control the process at the present time, k?

d) Assume that we have some output constraints

$$y_{k+1|L}^{min} \leq y_{k+1|L} \leq y_{k+1|L}^{max} \tag{7}$$

Show that the constraints can be written as a linear inequality

$$\mathcal{A}u_{k|L} \leq b \tag{8}$$

Specify the matrix $\mathcal{A}$ and the vector b.

e) The control objective, (4), with the prediction model (5) and the constraints (8) can be formulated as a Quadratic Programming (QP) problem,

Problem 0.1 (MPC QP problem)
Minimize

$$J_k = u_{k|L}^T H u_{k|L} + 2f^T u_{k|L} + J_0. \tag{9}$$

with respect to $u_{k|L}$ *subject to constraints as in (8).*

Specify matrix H and the vector f.

Task 2 (20%) (Computing present state, x_k)

Given a discrete time state space model

$$x_{k+1} = Ax_k + Bu_k, \tag{10}$$
$$y_k = Dx_k. \tag{11}$$

From the state space model (10) and (11) we can deduce the Prediction Model (PM)

$$y_{k+1|L} = O_L A\hat{x}_k + F_L u_{k|L}, \tag{12}$$

In order to use this PM in case when the state vector x_k is not measured, we have to obtain an estimate $\hat{x}_k$.

a) Show that the present state can be expressed by

$$x_k = A^{J-1}x_{k-J+1} + C_{J-1}^d u_{k-J+1|J-1} \tag{13}$$

where J is a horizon into the past. Give an expression for the matrix C_{J-1}^d and the vector $u_{k-J+1|J-1}$. Tips: this equation may be deduced from (10).

b) Consider the matrix equation

$$y_{k|J} = O_J x_k + H_J^d u_{k|J-1}. \tag{14}$$

Define the matrices O_J and H_J^d.

c) Use equations (13) and (14) in order to find an expression

$$\hat{x}_k = f_2(\text{past inputs and outputs}) \tag{15}$$

Specify the function $f_2(\cdot)$!

Task 3 (25%) (Discrete LQ optimal control with Integral Action)

We are in this task to study an LQ optimal controller for a system described by the state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v, & (16)\\ y_k &= Dx_k + w, & (17)\end{aligned}$$

where v and w are constant and unknown disturbances.
Subject to the above state space model we want to design MPC optimal controller which minimizes the following criterion

$$J_k = \frac{1}{2}\sum_{i=1}^{L}((y_{k+i} - r_{k+i})^T Q_i (y_{k+i} - r_{k+i}) + \Delta u_{k+i-1}^T P_i \Delta u_{k+i-1}). \tag{18}$$

where $\Delta u_k = u_k - u_{k-1}$ and r_k is a specified reference vector. Q_i and P_i are symmetric and positive semidefinite matrices.

a) Show that it is possible to write the model in (16) and (17) on deviation form, i.e.,

$$\begin{aligned} \Delta x_{k+1} &= A\Delta x_k + B\Delta u_k, & (19)\\ \Delta y_k &= D\Delta x_k, & (20)\end{aligned}$$

where

$$\Delta x_k = x_k - x_{k-1}, \;\; \Delta u_k = u_k - u_{k-1}, \;\; \Delta y_k = y_k - y_{k-1}. \tag{21}$$

What can be gained by doing this?

b) Show that the model in (19) and (20) can be written as follows

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \quad (22) \\ y_k &= \tilde{D}\tilde{x}_k, \quad (23) \end{aligned}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix}. \quad (24)$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) Find the optimal unconstrained control deviation

$$\Delta u^*_{k|L} = f_3(\cdot) \quad (25)$$

in this case ! Specify the function $f_3(\cdot)$ in this case.

d) Assume that we have some control rate of change constraints

$$\Delta u^{min}_{k|L} \leq \Delta u_{k|L} \leq \Delta u^{max}_{k|L} \quad (26)$$

Show that the constraints can be written as a linear inequality

$$\mathcal{A}\Delta u_{k|L} \leq b \quad (27)$$

Specify the matrix $\mathcal{A}$ and the vector b in this case.

e) Formulate the MPC problem as a QP problem for computing the optimal future control deviations $\Delta u^*_{k|L}$.