Final Exam Course SCE4106 Model Predictive Control with Implementation Friday December 9, 2011 kl. 9.00 - 12.00

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Task 1 (10%) (Linear quadratic problem)

Given an objective function

$$J = (y - r)^T Q(y - r) + u^T Pu, \tag{1}$$

where y, r and u are vectors and Q and P symmetric positive matrices of appropriate dimensions, and where

$$y = Fu + p. \tag{2}$$

a) Show that the objective (1) with model (2) may be written as

$$J = u^T Hu + 2f^T u + J_0. \tag{3}$$

Specify the matrix H, the vector f and the scalar parameter J_0 !

b) Show that the optimal solution may be written on the form

$$u^* = G(r - p). \tag{4}$$

Specify the feedback matrix G.

Task 2 (30%) (MPC with Integral Action)

Consider a system described by the discrete state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v, & (5)\\ y_k &= Dx_k + w, & (6)\end{aligned}$$

where v and w are possibly unknown constant disturbances.
We want to design an MPC controller which minimizes the following criterion

$$J_k = \sum_{i=1}^{L}((y_{k+i} - r_{k+i})^T Q_i(y_{k+i} - r_{k+i}) + \Delta u_{k+i-1}^T P_i \Delta u_{k+i-1}), \tag{7}$$

where $\Delta u_k = u_k - u_{k-1}$ and r_k is a specified reference vector. $Q_i \geq 0$ and $P_i \geq 0$ are weighting matrices.

a) Usually linear and linearized non-linear models of dynamic systems are of the form

$$\begin{aligned} \delta x_{k+1} &= A\delta x_k + B\delta u_k, & (8)\\ \delta y_k &= D\delta x_k, & (9)\end{aligned}$$

where the variables $\delta x_k = x_k - x_0$, $\delta u_k = u_k - u_0$ and $\delta y_k = y_k - y_0$ in the linear state space model (8) and (9) are deviation variables valid around some nominal steady state values x_0, u_0 and y_0.

Show that the linear model (8) and (9) may be written in terms of the actual physical variables x_k, u_k and y_k, and on the form as the linearized model in (5) and (6) ? Define v and w in this case !

b) The above MPC criterion, (7), may be written in more compact form as

$$J_i = (y_{k+1|L} - r_{k+1|L})^T Q (y_{k+1|L} - r_{k+1|L}) + \Delta u_{k|L}^T P \Delta u_{k|L}. \tag{10}$$

Specify the matrices Q and P, and the vectors $y_{k+1|L}$, $r_{k+1|L}$ and $\Delta u_{k|L}$ in the objective (10).

c)

1. Show that it is possible to write the model in (5) and (6) on deviation form, i.e.,

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \tag{11}$$
$$\Delta y_k = D\Delta x_k, \tag{12}$$

where

$$\Delta x_k = x_k - x_{k-1},\ \ \Delta u_k = u_k - u_{k-1},\ \ \Delta y_k = y_k - y_{k-1}. \tag{13}$$

What can be gained by doing this?

2. Show that the model in (11) and (12) can be written as follows

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, \tag{14}$$
$$y_k = \tilde{D}\tilde{x}_k, \tag{15}$$

where

$$\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix}. \tag{16}$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

d) From the augmented state space model (14) and (15) we define a Prediction Model (PM) of the form

$$y_{k+1|L} = p_L + F_L \Delta u_{k|L}. \tag{17}$$

Specify the matrix F_L and the vector p_L !

e) Show that the performance index (10) can be written as

$$J_k = \Delta u_{k|L}^T H \Delta u_{k|L} + 2 f_k^T \Delta u_{k|L} + J_0. \tag{18}$$

Specify the matrix H and the vector f_k !

f) Find the optimal unconstrained control deviation

$$\Delta u_{k|L}^* = f(\cdot), \tag{19}$$

and specify the function $f(\cdot)$! You should also specify the optimal MPC control, u_k^*, at present time k.

g) Assume that we have the following constraints

$$\begin{aligned} \Delta u_{k|L}^{min} \leq \Delta u_{k|L} &\leq \Delta u_{k|L}^{max}, & (20)\\ u_{k|L}^{min} \leq u_{k|L} &\leq u_{k|L}^{max}. & (21)\\ & & (22) \end{aligned}$$

- Show that the constraints can be written as a linear inequality

$$\mathcal{A} \Delta u_{k|L} \leq b. \tag{23}$$

  Specify the matrix $\mathcal{A}$ and the vector b in this case.

- How can we solve the constrained MPC problem ? i.e., minimizing the objective (18) with respect to $\Delta u_{k|L}$ subject to the linear inequality (23). Describe short with words.

Task 3 (15%) (Computing present state, x_k)

Given a discrete time state space model

$$\begin{aligned} x_{k+1} &= A x_k + B u_k, & (24)\\ y_k &= D x_k. & (25) \end{aligned}$$

From the model (24) and (25) we may construct a Prediction Model (PM) which is a function of the present state, x_k. In case when the state vector x_k is not measured, we may use an estimate $\hat{x}_k$.

a) From the model (24) and (25) we construct a PM

$$y_{k+1|L} = p_L + F_L u_{k|L}. \tag{26}$$

Define expressions for the terms p_L and F_L in Eq. (26).

b) Show that the present state can be expressed by

$$x_k = A^{J-1}x_{k-J+1} + C^d_{J-1}u_{k-J+1|J-1}, \tag{27}$$

where J is a horizon into the past. Give an expression for the matrix C^d_{J-1} and the vector $u_{k-J+1|J-1}$. Tips: this equation may be deduced from (24).

In particular define the terms in Eq. (27) for $J = 4$.

c)

- Consider the matrix equation

$$y_{k|J} = O_J x_k + H^d_J u_{k|J-1}. \tag{28}$$

  Define the matrices O_J and H^d_J in terms of the model matrices A, B and D. Use $J = 4$ as an example.
- Use equations (27) and (28) in order to find an estimate of the present state of the form

$$\hat{x}_k = K_y\tilde{y} + K_u\tilde{u}. \tag{29}$$

  Define the gain matrices K_y and K_u and the vectors $\tilde{y}$ and $\tilde{u}$!

Task 4 (15%) Diverse MPC related questions

a) A Finite Impulse Response (FIR) model

$$y_k = h_1u_{k-1} + h_2u_{k-2} + h_3u_{k-3}, \tag{30}$$

can be written as a state space model

$$x_{k+1} = Ax_k + Bu_k, \tag{31}$$
$$y_k = Dx_k. \tag{32}$$

Define the model matrices A, B and D.

b) Given a state space model

$$x_{k+1} = Ax_k + Bu_k, \tag{33}$$
$$y_k = Dx_k. \tag{34}$$

Define a Prediction Model (PM) of the form

$$y_{k+1|L} = F_L u_{k|L} + p_L, \tag{35}$$

from the state space model (33) and (34). Define the matrix F_L and the vector p_L where L is the prediction horizon.

c) Consider the FIR model in step 4a), i.e.,

$$y_k = h_1 u_{k-1} + h_2 u_{k-2} + h_3 u_{k-3}. \tag{36}$$

Chose a prediction horizon $L = 4$ and find a Prediction Model (PM) of the form

$$y_{k+1|4} = F_4 u_{k|4} + p_4, \tag{37}$$

directly from the FIR model (36). Define the matrix F_4 and the vector p_4 ? Tips: they are functions of the model parameters h_1, h_2 and h_3.