Final Exam Course SCE4106 Model Predictive Control with Implementation Monday 14th December 2009 kl. 9.00 - 12.00

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Task 1 (30%) (MPC with Integral Action)

Consider a system described by the discrete state space model

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + v, && (1)\\
y_k &= Dx_k + w, && (2)
\end{aligned}
$$

where v and w are possibly unknown constant disturbances.
We want to design an MPC controller which minimizes the following criterion

$$
J_k = \sum_{i=1}^{L} ((y_{k+i} - r_{k+i})^T Q_i (y_{k+i} - r_{k+i}) + \Delta u_{k+i-1}^T P_i \Delta u_{k+i-1}), \tag{3}
$$

where $\Delta u_k = u_k - u_{k-1}$ and r_k is a specified reference vector. $Q_i \geq 0$ and $P_i \geq 0$ are weighting matrices.

a) The above MPC criterion, (3), may be written in more compact form as

$$
J_i = (y_{k+1|L} - r_{k+1|L})^T Q (y_{k+1|L} - r_{k+1|L}) + \Delta u_{k|L}^T P \Delta u_{k|L}. \tag{4}
$$

Specify the matrices Q and P, and the vectors $y_{k+1|L}$, $r_{k+1|L}$ and $\Delta u_{k|L}$ in the objective (4).

b)

1. Show that it is possible to write the model in (1) and (2) on deviation form, i.e.,

$$
\begin{aligned}
\Delta x_{k+1} &= A\Delta x_k + B\Delta u_k, && (5)\\
\Delta y_k &= D\Delta x_k, && (6)
\end{aligned}
$$

where

$$
\Delta x_k = x_k - x_{k-1}, \;\; \Delta u_k = u_k - u_{k-1}, \;\; \Delta y_k = y_k - y_{k-1}. \tag{7}
$$

What can be gained by doing this?

2. Show that the model in (5) and (6) can be written as follows

$$
\begin{aligned}
\tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\Delta u_k, && (8)\\
y_k &= \tilde{D}\tilde{x}_k, && (9)
\end{aligned}
$$

where

$$
\tilde{x}_k = \begin{bmatrix} \Delta x_k \\ y_{k-1} \end{bmatrix}. \tag{10}
$$

Here you should define the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$.

c) From the augmented state space model (8) and (9) we define a Prediction Model (PM) of the form

$$y_{k+1|L} = p_L + F_L \Delta u_{k|L}. \tag{11}$$

Specify the matrix F_L and the vector p_L !

d) Show that the performance index (4) can be written as

$$J_k = \Delta u_{k|L}^T H \Delta u_{k|L} + 2 f_k^T \Delta u_{k|L} + J_0. \tag{12}$$

Specify the matrix H and the vector f_k !

e) Find the optimal unconstrained control deviation

$$\Delta u_{k|L}^* = f(\cdot), \tag{13}$$

and specify the function $f(\cdot)$! You should also specify the optimal MPC control, u_k^*, at present time k.

f) Assume that we have some control rate of change constraints

$$\Delta u_{k|L}^{min} \leq \Delta u_{k|L} \leq \Delta u_{k|L}^{max}. \tag{14}$$

Show that the constraints can be written as a linear inequality

$$\mathcal{A} \Delta u_{k|L} \leq b. \tag{15}$$

Specify the matrix $\mathcal{A}$ and the vector b in this case.

Task 2 (25%) (Computing present state, x_k, and state observer)

Given a discrete time state space model

$$x_{k+1} = A x_k + B u_k, \tag{16}$$
$$y_k = D x_k. \tag{17}$$

From the model (16) and (17) we may construct a Prediction Model (PM) which is a function of the present state, x_k. In case when the state vector x_k is not measured, we have to obtain an estimate $\hat{x}_k$.

a) From the model (16) and (17) we construct a PM

$$y_{k+1|L} = p_L + F_L u_{k|L}. \tag{18}$$

Give an expression for the term p_L in (18).

b) Show that the present state can be expressed by

$$x_k = A^{J-1} x_{k-J+1} + C^d_{J-1} u_{k-J+1|J-1}, \tag{19}$$

where J is a horizon into the past. Give an expression for the matrix C^d_{J-1} and the vector $u_{k-J+1|J-1}$. Tips: this equation may be deduced from (16).

c) Consider the matrix equation

$$y_{k|J} = O_J x_k + H^d_J u_{k|J-1}. \tag{20}$$

Define the matrices O_J and H^d_J.

d) Use equations (19) and (20) in order to find an expression

$$\hat{x}_k = f_2(\text{past inputs and outputs}). \tag{21}$$

Specify the function $f_2(\cdot)$!

e) Formulate an alternative state observer, e.g. Kalman filter, way of obtaining an estimate, $\hat{x}_k$, of the present state, x_k.

Task 3 (15%): Optimal control and PLS

Given a discrete time state space model

$$x_{k+1} = A x_k + B u_k, \tag{22}$$

$$y_k = D x_k, \tag{23}$$

where $k = 0, 1, 2, \ldots, a$ is discrete time. Consider an objective criterion

$$J = (r - y_a)^T Q (r - y_a). \tag{24}$$

a) Show that the state vector, x_k, in (22) at a discrete time $k = a \geq 1$ may be written on the form

$$x_a = A^k x_0 + C_a u_{0|a}, \tag{25}$$

where matrix C_a is a function of the model matrices A, B and D. Specify the matrix C_a and the vector $u_{0|a}$ of controls !

b) Assume that the initial state, $x_0 = 0$. Show that the objective criterion, (24), may be written as

$$J = u_{0|a}^T H u_{0|a} + 2 f^T u_{0|a} + J_0. \tag{26}$$

Specify the matrix H, the vector f and the scalar J_0 in the objective criterion (26) !

c) Find the optimal controls, $u^*_{0|a}$, and the corresponding optimal state vector x^*_a !

Epilog This task is motivated from the Partial Least Squares (PLS) regression problem, where one may show that the PLS problem may be formulated as an equivalent optimal control problem where the vector of PLS regression coefficients, $B_{PLS} = K_a(K_a^T X^T X K_a)^{-1} K_a^T X^T Y$, is equal to the optimal state x^*_a in the corresponding optimal control problem.