Master study
Systems and Control Engineering
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SCE4006 Model Predictive Control with Implementation

Exercise 4

Task 1

Basic literature for solving this exercise is found in the Lecture Notes. Especially see Sections 2.2, 2.3, 2.4.1, 3.3, and Examples 4.5 and 4.6.

a)

Consider given the Quadratic Programming (QP) problem

$$\begin{array}{l} \min_u J(u) = u^T H u + 2 f^T u, \\ \text{with constraints} \quad Au \le b, \end{array} \tag{1}$$

where

$$H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad f = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}. \tag{2}$$

The matrices in the inequality constraint are given by

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}. \tag{3}$$

- Find the solution to the corresponding unconstrained QP problem?
- Solve the QP problem with the active set method, Lecture Notes Ch. 3.3.2. Use MATLAB for the numerical calculations. Check your answer by using the MATLAB function **quadprog** for solving QP problems.

b) Let us study Example 5.6 at page 41 in the Lecture notes. Here we have a prediction horizon of $L = 2$.

We demand that the control u_k should be constrained as follows

$$0 \le u_k \le 0.2. \tag{4}$$

Furthermore, we are specifying that there is a change in the reference signal r_{k+1} from 0 to 1 at time instant $k = 4$ and that the future reference signal is given by

$$r_{k+1|2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \forall \ k = 1, 2, 3 \tag{5}$$

$$r_{k+1|2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \forall \ k = 4, 5, 6, \ldots \tag{6}$$

Formulate the QP problem the MPC controller (EMPC) solves at time instant $k = 4$. Use can use the initial value $x_4 = y_4 = \frac{1}{2}$. You should find that this problem is identical to the QP problem in Step a) above.

c) Study Example 4.6 in the Lecture Notes in detail. You can use the MATLAB script file **main_ex46** as the starting point. Perform simulations of the MPC controlled system for varying weighting parameters q and p in the optimization criterion.

$$\begin{aligned} J_k &= \sum_{i=1}^{2}(q(y_{k+i} - r_{k+i})^2 + pu_{k+i-1}^2) \\ &= (y_{k+1|2} - r_{k+1|2})^T Q(y_{k+1|2} - r_{k+1|2}) + u_{k|2}^T P u_{k|2}, \\ &= u_{k|2}^T H u_{k|2} + 2f^T u_{k|2} + J_0. \end{aligned} \tag{7}$$

Note here that the weighting matrices are simply given by $Q = qI_2$ and $P = pI_2$ where q and p are scalar weighting parameters.

Task 2

Consider the QP problem

$$J = 2x_1^2 + x_1 x_2 + x_2^2 - 12x_1 - 10x_2 \tag{8}$$

subject to the constraints

$$x_1 + x_2 \leq 4 \tag{9}$$

$$x_1 \geq 0 \tag{10}$$

$$x_2 \leq 0.5 \tag{11}$$

a) Solve the unconstraind QP problem, i.e., find the minimum of (8) with respect to x_1 and x_2.

b) Formulate the QP problem (8) and (9)-(11), as a QP problem with inequality constraints

$$J = x^T H x + 2f^T x \tag{12}$$

subject to

$$Ax \leq b \tag{13}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

Find the the matrices H, f, A and b such that the problems are equivalent.

c) Solve the QP problem with inequality constraints in step 2b) above with an active set method.

Task 3

Consider a system with a pure integrator

$$y_{k+1} = y_k + u_k \tag{15}$$

and a MPC criterion

$$J_k = \sum_{i=1}^{2}((y_{k+i} - r_{k+i})^2 + u_{k+i-1}^2) \tag{16}$$

We want to control the system with a standard MPC algorithm.

a) Show that the control objective may be written as

$$J_k = u_{k|2}^T H u_{k|2} + 2 * f_k^T u_{k|2} + J_0 \tag{17}$$

Specify in particular the matrix H and the vector f_k.

b) Simulate the unconstrained MPC control. Use a binary signal reference generated by

```
N=500;
R=prbs1(N,40,80);
```

c) Introduce constraints

$$-u_{min} \leq u_k \leq u_{max} \tag{18}$$

Formulate the constraints as an inequality

$$\mathcal{A}u_{k|2} \leq b \tag{19}$$

specify $\mathcal{A}$ and b.

d) Simulate the constrained MPC problem by using the MATLAB quadprog function.

Some tips and solutions for Task 1

The MPC controller will at each time instant solve the following QP problem

$$J_k(u_{k|L}) = u_{k|L}^T H u_{k|L} + 2 f_k^T u_{k|L}, \tag{20}$$

with inequality constraints

$$A u_{k|L} \leq b \tag{21}$$

Furthermore we have that

$$\begin{aligned} H &= F_L^T Q F_L + P, & (22)\\ f_k &= F_L^T Q (p_L - r_{k+1|L}), & (23)\\ p_L &= O_L A x_k. & (24) \end{aligned}$$

At time instant $k = 4$ we have that

$$f_4 = F_2^T Q O_2 A x_4 - F_2^T Q r_{5|2} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}. \tag{25}$$

and

$$H = F_2^T Q F_2 + P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{26}$$

The matrices F_2, O_2, P and Q are as presented in Example 4.6 at page 31 in the Lecture Notes.

1 MATLAB scripts

```
% Solution proposal for Exercise 4, task 1a).
% Solving a constrained QP problem with the Active set method

% Active set method:
%  1. if the corresponding Lagrangian multiplier, li <0, then the
%  ith inequality, ai'*u \leq bi, is inactive. Hence, put li=0.
%  2. if the langrange multiplier is positive, li>=0, then the inequality
%  is active.
%  3. Store all, m, Langrange multipliers, li for i=1,2,..,m in a vector.
%  and compute the solution.

% Task a)
% Given matrices in QP problem with inequality constraints
H=[3,1;1,2]; f=[-1;-0.5];
A=[1,0;-1,0]; b=[0.2;0];

% 1) Checking the first inequality, it it is actice ore not.
a1=A(1,:)'; b1=b(1);
l_1=-(b1+a1'*inv(H)*f)/(a1'*inv(H)*a1)
if l_1 <=0
   l_1=0;
end


% 2)  Cheking the 2nd inequality, if it is actice ore not.
a2=A(2,:)'; b2=b(2);
l_2=-(b2+a2'*inv(H)*f)/(a2'*inv(H)*a2)
if l_2 <=0
   l_2=0;
end

% The solution to the constrained QP problem.
l=[l_1;l_2]
u=-inv(H)*(f+A'*l)

% Cecking the results with the MATLAB function quadprog.
u_qp=quadprog(H,f,A,b)
```

```
% main_ex46.m
% Matrix QP implementation of example 4.6.
% DDIR, 15.03.2001
clear all

A=1; B=1; D=1; q=1; p=1;          % process model matrices and weighting parameters.

F=[D*B 0;D*A*B D*B];              % EMPC matrices.
O_L=[D;D*A];
Q=eye(2)*q; P=eye(2)*p;
H=F'*Q*F+P;

Ac=[1,0;-1,0]; bc=[0.2;0];        % Describing the inequality constraint, u_k <= 0.2

t=0:1:10; N=length(t);            % Time intervall.
x=0.5;                            % Initial value for the state.
for i=1:N
   if i < 4                       % Specified reference.
      r=[0.5;0.5];
   elseif i>=4;
      r=[1;1];
   end
   y=x;                           % Process measurement.

   p=O_L*A*x;
   f=F'*Q*(p-r);
   uf=quadprog(H,f,Ac,bc);        % Equivalent unconstrained controls, uf=-inv(H)*f;
   u=uf(1);                       % Only 1st variable is used for control.
   U(i,:)=uf'; Y(i,1)=y;          % Store y and u.
   x=x+u;                         % Update state equation (put u to the process).
end
subplot(211), plot(t,U(:,1),'-',t,U(:,1),'bo'),grid, ylabel('u_k'),axis([0,10,-0.1,0.4
title('Simulation of the MPC system in Example 4.6')
subplot(212), plot(t,Y,'-',t,Y,'bo'), grid, ylabel('y_k'), axis([0,10,0.4,1.1])
xlabel('Discrete time')
```