

Master study
Systems and Control Engineering
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SCEV3006 Advanced Control with Implementation

Solution Exercise 1

The system can be written in matrix state space form as follows

$$\dot{x} = \overbrace{\begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix}}^A x + \overbrace{\begin{bmatrix} \beta \\ 1 \end{bmatrix}}^B u, \quad (1)$$

$$y = \underbrace{\begin{bmatrix} 1 & \delta \end{bmatrix}}_D x. \quad (2)$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution task 1b)

The controllability matrix for this system is given by

$$C_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \beta & -5\beta - 2 \\ 1 & -1 \end{bmatrix} \quad (3)$$

The system is controllable if $\text{rank}(C_2) = 2$. This can be investigated in at least two ways.

Method 1 No columns or rows in C_n should be linearly dependent in order for the controllability matrix to have full rank, i.e., $\text{rank}(C_2) = 2$ since $n = 2$ in this example. This means that there should not exist a constant k such that

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} = k \begin{bmatrix} -5\beta - 2 \\ -1 \end{bmatrix} \quad (4)$$

Since there are two unknowns in the problem we chose k , e.g., in such a way that the left and right hand sides becomes as close as possible. $k = -1$ gives

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} = \begin{bmatrix} 5\beta + 2 \\ 1 \end{bmatrix} \quad (5)$$

We see that the two vectors becomes equal when $5\beta + 2 = \beta$, and that the two vectors always is different, i.e. linearly independent, when

$$5\beta + 2 \neq \beta \quad (6)$$

which shows that the system is controllable if $\beta \neq -\frac{1}{2}$.

Method 2 A simple method is to investigate the determinant of C_2 . The system is controllable if

$$\det(C_2) = -\beta + 5\beta + 2 \neq 0 \quad (7)$$

This gives that $\beta \neq -\frac{1}{2}$ for the system to be controllable.

Note that the demand $\text{rank}(C_n) = n$ only is equivalent to investigate if $\det(C_n) \neq 0$ for systems with a single control input variable, i.e. for $r = 1$. The determinant method can not be used for controllability analysis for systems with multiple control input variables, i.e. when $r > 1$, because C_n is not a quadratic matrix in this case.

Solution task 1c)

The observability matrix of the system is given by

$$O_2 = \begin{bmatrix} D \\ DA \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -5 & -2 - \delta \end{bmatrix} \quad (8)$$

The system is observable if $\text{rank}(O_2) = 2$ which for systems with one output variable is equivalent to check whether

$$\det(O_2) = -2 - \delta + 5\delta \neq 0 \quad (9)$$

This means that the system is observable if $\delta \neq \frac{1}{2}$.

Solution task 1d)

The diagonal form of a state space model $\dot{x} = Ax + Bu$, $y = Dx$, may be found by using a transformation $x = Mz$ where M is the eigenvector matrix to system matrix A and z is the state vector the diagonal (state space) form. This gives

$$\dot{z} = \Lambda z + M^{-1}Bu, \quad (10)$$

$$y = DMz, \quad (11)$$

where Λ is the diagonal eigenvalue matrix of the system

$$\Lambda = M^{-1}AM = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}. \quad (12)$$

An eigenvector matrix M (and the inverse) for this system is given by

$$M = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \quad (13)$$

We may now analyze controllability by studying the elements in the transformed matrix $M^{-1}B$ and we may study observability by checking the elements in the matrix DM .

The system is controllable if no row or rows in the matrix $M^{-1}B$ have all elements identically equal to zero. We have

$$M^{-1}B = \begin{bmatrix} \beta + \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}. \quad (14)$$

We see that the system is controllable if now columns in the matrix DM have all elements equal to zero.

We find that the system is controllable if $\beta \neq -\frac{1}{2}$

Furthermore we have that the system is observable if no columns in the matrix DM have all elements equal to zero. We have that

$$DM = \begin{bmatrix} 1 & 1 - 2\delta \end{bmatrix}. \quad (15)$$

As we see the system is observable if $1 - 2\delta \neq 0$, i.e. observable for $\delta \neq \frac{1}{2}$.

Solution to task 1e)

Given the system

$$\dot{x}_1 = -5x_1 - 2x_2 + \beta u \quad (16)$$

$$\dot{x}_2 = -x_2 + u \quad (17)$$

$$y = x_1 + \delta x_2 \quad (18)$$

We have the following state space model

$$\dot{x} = Ax + Bu \quad (19)$$

$$y = Dx + Eu \quad (20)$$

where the system matrices are given by

$$A = \begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} \beta \\ 1 \end{bmatrix} \quad (21)$$

$$D = \begin{bmatrix} 1 & \delta \end{bmatrix} \quad E = 0 \quad (22)$$

Controllability Gramian matrix

The system is stable because A has eigenvalues with negative real part, i.e., $\lambda_1 = -5$ and $\lambda_2 = -1$). The infinity time controllability Gramian matrix, W_c , then satisfy the Lyapunov matrix equation

$$AW_c + W_cA^T = -BB^T \quad (23)$$

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$$AW_c = \begin{bmatrix} -5 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} -5w_{11} - 2w_{21} & -5w_{21} - 2w_{22} \\ -w_{21} & -w_{22} \end{bmatrix} \quad (24)$$

$$W_cA^T = (AW_c)^T = \begin{bmatrix} -5w_{11} - 2w_{21} & -w_{21} \\ -5w_{21} - 2w_{22} & -w_{22} \end{bmatrix} \quad (25)$$

$$BB^T = \begin{bmatrix} \beta \\ 1 \end{bmatrix} \begin{bmatrix} \beta & 1 \end{bmatrix} = \begin{bmatrix} \beta^2 & \beta \\ \beta & 1 \end{bmatrix} \quad (26)$$

Putting (24), (25) and (26) in the Lyapunov equation (23) and we get the following three equations for the unknown elements in the matrix W_c , i.e.,

$$-10w_{11} - 4w_{21} = -\beta^2 \quad (27)$$

$$-6w_{21} - 2w_{22} = -\beta \quad (28)$$

$$-2w_{22} = -1 \quad (29)$$

This gives the controllability Gramian matrix

$$W_c = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{10}(\beta^2 - \frac{2}{3}\beta + \frac{2}{3}) & \frac{1}{6}(\beta - 1) \\ \frac{1}{6}(\beta - 1) & \frac{1}{2} \end{bmatrix} \quad (30)$$

In order for the system to be controllable we have to ensure that W_c is positive definite, i.e.,

$$W_c > 0 \quad (31)$$

This is ensured if the eigenvalues of W_c is all positive or if all sub determinants of W_c is positive. We have the following requirements (positive sub determinants of order 1 and 2)

$$w_{11} = \frac{1}{10}(\beta^2 - \frac{2}{3}\beta + \frac{2}{3}) > 0 \quad (32)$$

$$\det(W_c) = w_{11}w_{22} - w_{21}^2 = \frac{1}{45}(\beta^2 + \beta + \frac{1}{4}) > 0 \quad (33)$$

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$$\det(W_c) > 0 \quad \text{for} \quad \beta \neq -\frac{1}{2} \quad (34)$$

$w_{11} > 0$ for all real β .

We se that $w_{11} \rightarrow \infty$ når $\beta \rightarrow \infty$ and $\beta \rightarrow -\infty$. Furthermore we have that w_{11} have a minimum for $\frac{dw_{11}}{d\beta} = \frac{1}{3}$. This means that $w_{11} > 0$ for all real β . Note

however that $w_{11} = 0$ for $\beta = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$.

The conclusion is that the system is controllable for all $\beta \neq -\frac{1}{2}$.

Observability Gramian matrix

We solve the Lyapunov matrix equation

$$A^T W_o + W_o A = -D^T D \quad (35)$$

with respect to the observability Gramian matrix W_o . We have that

$$A^T W_o = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} -5w_{11} & -5w_{21} \\ -2w_{11} - w_{21} & -2w_{21} - w_{22} \end{bmatrix} \quad (36)$$

$$W_o A = (A^T W_o)^T = \begin{bmatrix} -5w_{11} & -2w_{11} - w_{21} \\ -5w_{21} & -2w_{21} - w_{22} \end{bmatrix} \quad (37)$$

$$D^T D = \begin{bmatrix} 1 \\ \delta \end{bmatrix} \begin{bmatrix} 1 & \delta \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \delta & \delta^2 \end{bmatrix} \quad (38)$$

This gives three equations

$$-10w_{11} = -1 \quad (39)$$

$$-2w_{11} - 6w_{21} = -\delta \quad (40)$$

$$-4w_{21} - 2w_{22} = -\delta^2 \quad (41)$$

which may be solved with respect to w_{11} , w_{21} and w_{22} . This gives

$$w_{11} = \frac{1}{10} \quad (42)$$

$$w_{21} = \frac{1}{6}(\delta - \frac{1}{5}) \quad (43)$$

$$w_{22} = \frac{1}{2}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) \quad (44)$$

This gives the observability Gramian matrix

$$W_o = \begin{bmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{6}(\delta - \frac{1}{5}) \\ \frac{1}{6}(\delta - \frac{1}{5}) & \frac{1}{2}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) \end{bmatrix} \quad (45)$$

We investigate if W_o may have rank less than two for some values of parameter δ . We have

$$\det(W_o) = w_{11}w_{22} - w_{21}^2 = \frac{1}{20}(\delta^2 - \frac{2}{3}\delta + \frac{4}{30}) - \frac{1}{36}(\delta - \frac{1}{5})^2 \quad (46)$$

$$= \frac{1}{45}(\delta^2 - \delta + \frac{1}{4}) = \frac{1}{45}(\delta - \frac{1}{2})^2 \quad (47)$$

We see that $\det(W_o) = 0$ for $\delta = \frac{1}{2}$. W_o is therefore singular (not invertable) for this value of δ . This means that the system is not observable for $\delta = \frac{1}{2}$.

Solution of Task 2

a) We try with a vector

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (48)$$

The controllability matrices and the coefficients in the characteristic polynomial are

$$p(A) = \lambda^4 + p_4\lambda^3 + p_3\lambda^2 + p_2\lambda + p_1 \quad (49)$$

For the two cases we have

Case 1

$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 1 & -5 & 23 & -65 \\ 0 & 0 & 12 & -48 \\ 0 & 2 & 2 & -22 \\ 0 & 1 & -1 & -5 \end{bmatrix} \quad (50)$$

and

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = -C_4^{-1}A^4b = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad (51)$$

Case 2

$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 1 & -3 & -1 & 11 \\ 0 & 10 & -8 & -6 \\ 0 & -12 & 20 & -24 \\ 0 & 4 & -8 & 12 \end{bmatrix} \quad (52)$$

It turns out that for this choice of b then C_4 may not be inverted, this means that the pair (A, b) is not controllable because $\text{rank}(C_4) = 3 < n = 4$, However choosing

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (53)$$

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$$C_4 = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} = \begin{bmatrix} 0 & -1 & -14 & 51 \\ 1 & 17 & -9 & -37 \\ 0 & -32 & 64 & -92 \\ 0 & 12 & -28 & 48 \end{bmatrix} \quad (54)$$

and

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = -C_4^{-1}A^4b = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \end{bmatrix} \quad (55)$$

b) A controllability canonical form is obtained by transforming the model with

$$z = C_4^{-1}x \quad (56)$$

i.e.,

$$\dot{z} = A_{co}z \quad (57)$$

where

$$A_{co} = C_4^{-1}AC_4 \quad (58)$$

with initial state $z(t=0) = C_4^{-1}x(t=0)$.

case 1 The system matrix on controllability canonical form is in this case

$$A_{co} = \begin{bmatrix} 0 & 0 & 0 & -p_1 \\ 1 & 0 & 0 & -p_2 \\ 0 & 1 & 0 & -p_3 \\ 0 & 0 & 1 & -p_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad (59)$$

Case 2 The system matrix on controllability canonical form is in this case

$$A_{co} = \begin{bmatrix} 0 & 0 & 0 & -p_1 \\ 1 & 0 & 0 & -p_2 \\ 0 & 1 & 0 & -p_3 \\ 0 & 0 & 1 & -p_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad (60)$$

Note that the controllability canonical form may be computed by first generating the controllability matrix

$$C_{n+1} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b & A^n b \end{bmatrix} \quad (61)$$

and thereafter compute

$$A_{co} = -C_{n+1}^{-1}(:, 1:n)C_{n+1}(:, 2:n+1); \quad (62)$$