Master study
Systems and Control Engineering
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# SCEV3006 Advanced Control with Implementation

## Exercise 4

## Task 1

A mathematical model of two inverted pendulums which stands below each other on a chart is given by

$$\ddot{q}_1 = \frac{m_1 l_{G1} g \sin(q_1)}{J_{t1} + m_1 l_{G1}^2} - \frac{m_1 l_{G1} \cos(q_1)}{J_{t1} + m_1 l_{G1}^2} \dot{v} \tag{1}$$

$$\ddot{q}_2 = \frac{m_2 l_{G2} g \sin(q_2)}{J_{t2} + m_2 l_{G2}^2} - \frac{m_2 l_{G2} \cos(q_2)}{J_{t2} + m_2 l_{G2}^2} \dot{v} \tag{2}$$

where $q_1$ is the angle for pendulum number 1, i.e. the angle of pendulum 1 from the vertical line. $q_2$ is the angle for pendulum number 2 and $v$ is the velocity of the chart.
The constants in the model is given by:

- $m_1 = 0.244$ is the mass of pendulum nr. 1 (inc. bar and load),
- $m_2 = 0.225$ is the mass of pendulum nr. 2 (inc. bar and load),
- $l_{G1} = 0.6369$ is the distance from origo to the center of gravity $G1$ for pendulum nr. 1.
- $l_{G2} = 0.3778$ is the distance from the holding on the chart to the point of gravity $G2$ for pendulum nr. 2.
- $J_{t1} = 0.0062$ $[kgm^2]$ is the moment of inertia for bar 1 with load about the center of gravity $G1$.
- $J_{t2} = 0.0012$ $[kgm^2]$ is the moment of inertia for bar 2 with load about the center of gravity $G2$.
- $g = 9.81$ is the acceleration of the gravity.

## Part 1

We will in this part use a simplified model for the control actuator, i.e. the chart, such that $M\dot{v} = u$ where $M = 1$ is the mass of the chart and $u$ is the manipulable force (control) on the chart.
Sketch a figure for the system.

**a)** Use the following state definitions

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} \tag{3}$$

and write up a state space model as a set of 1st order differential equations. Do not insert numerical values at this stage.

**b)** Use the state vector defined in step **a)** above and find a linearized state space model of the form

$$\dot{x} \quad = \quad Ax + Bu \tag{4}$$

**c)** Put numerical values into the linearized state space model and analyze the system for stability, controllability and observability. We are assuming that both $x_1 = q_1$ and $x_2 = q_2$ are measured (the angle velocities are not measured).

**d)** Compute the LQ optimal controller for the system and simulate the closed loop system with different initial values, i.e., $x_1(t = 0) = q_1(t = 0) \neq 0$ and $x_2(t = 0) = q_2(t = 0) \neq 0$. You can here assume that all states are measured and available for feedback control. Study both the linear and the non-linear system under LQ optimal control feedback.

## Part 2

We will in this part of the exercise study two different alternative equations of motion for the chart.
**Alternative 2**. An alternative model for the control actuator is given by

$$M\dot{v} = -m_1 g \tan(q_1) - m_2 g \tan(q_2) + u \tag{5}$$

This model can be deduced from a simplified force balance on the chart in the horizonal plane. How will this model influence upon the results in Part 1 of the exercise?
**Alternative 3**. A third alternative equation of motion for the chart with the two inverted pendulums is given by

$$M\dot{v} = u - m_1 a_{x1} - m_2 a_{x2} - bv \tag{6}$$

where $a_{x1}$ and $a_{x2}$ is the acceleration components of the pendulums in the direction of movement (i.e. the x-axis), given by

$$a_{x1} = \dot{v} + l_{G1}\cos(q_1)\ddot{q}_1 - l_{G1}\sin(q_1)\dot{q}_1^2 \quad (7)$$
$$a_{x2} = \dot{v} + l_{G2}\cos(q_2)\ddot{q}_2 - l_{G2}\sin(q_2)\dot{q}_2^2 \quad (8)$$

How should the problem now be solved?

## Task 2

Given a system

$$\dot{x} = Ax + Bu \tag{9}$$
$$\tag{10}$$

where

$$A = \begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

We want to control the system with an LQ optimal controller which minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T P u) dt \tag{12}$$

where

$$Q = \begin{bmatrix} 135 & 60 \\ 60 & 135 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

**a)** Write down an expression for the optimal control. The solution is a function of the matrix $R$ which is found as the positive definite solution to a Riccati equation, ore via the transition matrix (matrix exponential) of the Hamiltonian matrix $F$. Write down the equations for both alternatives...

**b)** Put into numerical values and find the LQ optimal controller. Use MATLAB for the computations, and perform both alternatives sketched in step 2a) above.

**c)** Compute the eigenvalues of the closed loop system. Can you say something about the eigenvalues of the closed loop system compared to the open loop system?