Master study
Systems and Control Engineering
Department of Technology
Telemark University College
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SCEV3006 Advanced Control with Implementation

Exercise 5

Task 1

Given a process described by the following model

$$\dot{x} = ax + bu + cv \tag{1}$$
$$y = x \tag{2}$$

Consider given a set-point y_0 for the output y.

1. The process is to be controlled by a PI controller and a feed forward from the disturbance v. Sketch a block diagram for the controlled system in the time domain.
2. How would you design a conventional PI controller for this process?
3. Design an LQ optimal PI controller for the process. Assume $v = 0$ and $y_0 = 0$ under the design procedure.
4. Assume now that v is measured and that it is slowly varying. Design an LQ optimal feed forward controller from the disturbance v. Tips: augment the process model with a model for the disturbance and formulate the LQ optimal control problem for the extended process.

Task 2

Given a process described by the state space model

$$\dot{x} = Ax + Bu \tag{3}$$

and the criterion

$$J = \int_0^{t_1} (x^T Q x + u^T P u) dt \tag{4}$$

which is to be minimized with respect to the control u. We are letting $t_1 = \infty$.

1. Use the maximum principle and develop the meaning of the Hamiltonian matrix

$$\mathrm{F} = \begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \tag{5}$$

What is the expression for the matrix H. Tips: you may work with a modified criterion which is equal to the above criterion, J, multiplied with $\frac{1}{2}$.

2. Given a block real Schur decomposition of the Hamiltonian matrix. See for example the MATLAB functions **schur2** and **blkrsch2**.

$$F = UTU^T \tag{6}$$

where U is an orthogonal matrix, i.e., $UU^T = I$ and $U^T = U^{-1}$. We have

$$\begin{bmatrix} A & -H \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \tag{7}$$

and such that the n stable eigenvalues in F is collected in T_{11}. Show that

$$R = U_{21}U_{11}^{-1} \tag{8}$$

is a solution to the Algebraic Riccati equation (ARE)

$$A^T R + RA - RHR + Q = 0 \tag{9}$$

3. Find an expression for the eigenvalues of the Hamiltonian matrix F. Tips: n eigenvalues are given by $A - HR$,

4. What are the demands for the system matrices A, B and weighting matrices Q and P in order for the LQ optimal system to be stable.

5. How many solutions does the Riccati equation have? Which of the solutions is used in the LQ optimal feedback?

6. Given the system and the weighting matrices

$$A = \begin{bmatrix} -0.5 & 0 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{10}$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = 1 \tag{11}$$

use the Schur method in order to find the positive solution to the continuous time Algebraic Riccati Equation (ARE). Tips: this task have to be solved numerically. The MATLAB function **blkrsch2.m** can be used to compute the block real Schur decomposition. This function can be downloaded or copied from the info-disc under the directory avreg\oving5\are_schur.

7. May the Schur method above be used in order to solve the time varying Riccati differential equation ?, i.e., in case when the time horizon t_1 in the criterion is finite.

Task 3

Tips: The basic theory used in this task is presented in section 4.12 "Analytical solution to the scalar LQ problem" in the Lecture notes.
Vi ønsker å optimalregulere temperaturen i et rom. La θ være temperaturen i rommet, θ_a være omgivelses-temperaturen og u være tilført effekt.
En modell for romtemperaturen er

$$\dot{\theta} = a(\theta - \theta_a) + bu. \tag{12}$$

der $a = -\frac{1}{T_c}$, $T_c = 5$ [min] og $b = 1$. Vi har at $\theta_a = 10$ [$^\circ C$] og at initialtemperaturen i rommet er $\theta(t_0) = \theta_a$.
Vi ønsker å regulere rommet slik at romtemperaturen er nær en ønsket temperatur $\theta_d = 20$ [$^\circ C$] ved tiden $t_1 = 60$ [min]. I tillegg ønsker vi å benytte minst mulig energi. Vi definerer følgende objektfunksjon

$$J = \frac{1}{2}s(\theta - \theta_d)^2 + \frac{1}{2}\int_{t_0}^{t_1} pu^2 dt, \tag{13}$$

der $p = 1$ og $s = 2$.

a) Definer tilstanden $x = \theta - \theta_a$ og formuler optimalreguleringsproblemet. Ikke sett inn tallverdier.

b) Finn løsningen på optimalreguleringsproblemet. Det skal finnes et analytisk uttrykk for $u(t)$ som funksjon av $x(t)$, x_r og enkelte av parametrene beskrevet over.

c) Sett inn tallverdier og simuler løsningen i MATLAB. Velg for enkelthets skyld $t_0 = 0$. Plot tidsforløp for $x(t)$, $u(t)$, $p(t)$ og $r(t)$. Tips: benytt Euler for å integerere prosessmodellen $\dot{x} = ax + bu$, benytt samplingsintervall $\Delta t = 0.1$.

d) Finn et analytisk uttrykk for $x(t_1)$, $u(t_1)$ og $p(t_1)$ og beregn de aktuelle verdiene. Sammenlign med verdiene i punkt c).

e) Anta nå at $u(t) = u(t_1)$ for $t > t_1$. Bestem den vekt s av slutt-tilstanden $x(t_1)$ som fører til at romtemperaturen svinger seg inn mot x_r. Dvs. bestem s slik at

$$\lim_{t \to \infty} x(t) = x_r. \tag{14}$$

Simuler systemet med denne vektingen.

f) Sett $t_0 = t$ og $t_1 = t + T$ der T er en konstant tidshorisont. Diskuter den optimale løsningen. Man skal her diskutere formen på det optimale pådraget $u(t)$. Bestem videre den stasjonære tilstanden

$$x_s = \lim_{t \to \infty} x(t) \tag{15}$$

som funksjon av horisonten T og vekten s. Dvs. forsøk å skissere $x_s = x_s(T, s)$.

Tips: Les avsnittet i kompendiet om den analytisk løsningen av optimalreguleringsproblemet for skalare systemer.

A MATLAB script for the solution of the Riccati equation with the Schur-method

```
function R=are_schur(A,B,Q,P)
% ARE_SCHUR Solve the continuous Algebraic Riccati Equation (ARE)
% with the block real schur decomposition of the Hamiltonian matrix.
% SYNTAX
% R=are_schur(A,B,Q,P)
% ON INPUT
% A,B - system matrices
% Q   - state weighting matrix
% P   - control weighting matrix
% ON OUTPUT
% R   - The positive definite solution to the ARE, A'R+RA-RBinv(P)B'R+Q=0.
% NON STANDARD MATLAB FUNCTIONS CALLED:
% BLKRSCH2, CSCHUR2
% Written by David Di Ruscio

F=[ A -B*pinv(P)*B';-Q -A'];           % Hamilton matrix
[U,T]=blkrsch2(F,1);                   % stable eigenvalues in upper
                                       % n x n submatrix T_11.
n=length(A);
U11=U(1:n,1:n);
U21=U(n+1:2*n,1:n);
R=U21*pinv(U11);
```

```
% Loesning til oving 5 oppgave 2 punkt 6.
% DDiR, 9. oktober 2000.
A=[-0.5 0;-1 0];
B=[1;0];
Q=[2 0;0 1];
P=1;

R=are_schur(A,B,Q,P)
G=-inv(P)*B'*R;

% TILLEGG: SIMULERING AV SYSTEMET.
x =[1;0.5];              % Initialverd for tilstandsvektoren.
yr=1.0;                  % Referansesignal.

dt=0.05;                 % Skrittlengde for Euler diskretisering.
t1=25;
t=0:dt:t1;
N=length(t);
for i=1:N                % Start simuleringslokke.
   y(i,:)=x';            % Lafrer tilstandene
   u=G*x-yr;             % regulator
   dotx=A*x+B*u;         % simulerer systemet med eksplisitt Euler integrasjon.
   x=x+dt*dotx;
end

plot(t,y);
xlabel('0 \leq t \leq 25')
grid
title('Simulering av LQ optimalregulert system')
```