Proposed solution MPC final exam 2023

This exam counts 60% for the final grade.

Problem 1

Tasks:

i) [15%] For practical implementation, which one of the two controller design will you choose even though all 4 states are measurable? Why? Explain with proper justifications.

Proposed Solution:

It would be necessary or useful to choose the output feedback MPC for implementation on a real system. Output feedback MPC contains a state estimator. At first it may seem unnecessary to have a state estimator since all the states are already measured using sensors. However, there are two important reasons as to why for a practical implementation it may become necessary to have a state estimator (like a kalman filter).

(a) For handling sensor failure situation: In circumstances where one or more of the sensors that are used for measuring the states fail, then the apriori state estimate obtained by the Kalman filter can be continued to be fed to the optimal control problem of MPC algorithm, without having to stop the MPC loop immediately. This may work out for some time before the close loop performance starts degrading significantly, which may be enough time for manual intervention to take place. If there was no kalman filter (state estimator), then a sensor failure situation would produce no measurement at all, and the MPC loop will immediately fail, and the closed loop response may become unstable.

(b) For filtering out measurement noise: Another advantage of having a state estimator or Kalman filter is to act as a low pass filter allowing the measurement noises to be filtered out. Direct noisy measurements fed into the MPC will produce noisy control signals, and it may also affect the proper functioning of the optimization solver used for solving the MPC problem, usually in the form of increased computational time.

ii) [5%] Explain briefly the difference between these two controller designs in general.

Suggested solution:

A state feedback MPC as the name implies is a controller where the states are directly fed back to the controller. This is only possible if all the states of the process being controlled are available or measurable i.e. full state information is a necessity. This makes the state feedback MPC an ideal case. However, in practice, it may not always be possible to measure all the state of the system. In such cases, the states of the system should be estimated. Estimation of the states can be performed by utilizing the available measurements. So in this case, the measurements (output) of the process is fed to an estimator (say a Kalman filter) which instead estimates the states. These estimated states are then fed to the MPC.

Problem 2

Tasks

i) [15%] Considering that the disturbance acting on the system is constant or slowly varying, and both the system states and the disturbances are perfectly known or measured, how will you formulate an LQ optimal control problem for achieving integral action?

Suggested solution:

The linear state space model of the process affected by disturbance is given as,

$$x_{k+1} = Ax_k + Bu_k + B_d d_k \tag{8.19}$$

$$y_k = Cx_k + C_d d_k \tag{8.20}$$

Now the process model of Equation 8.19 and 8.20 can be augmented with the disturbance model of Equation 8.18 as,

$$\underbrace{\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix}}_{\tilde{x}_{k+1}} = \underbrace{\begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_k \\ d_k \end{bmatrix}}_{\tilde{x}_k} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u_k \tag{8.21}$$

$$y_k = \underbrace{\begin{bmatrix} C & C_d \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} x_k \\ d_k \end{bmatrix}}_{\tilde{x}_k} \tag{8.22}$$

The augmented model is in a standard linear state space form as,

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k \tag{8.23}$$

$$y_k = \tilde{C}\tilde{x}_k \tag{8.24}$$

Here, $d_k \in \mathbb{R}^{n_d}$ with $n_d = n_y$ being the number of unmeasured disturbance variables and equal to the number of available measurement. The matrices $B_d \in \mathbb{R}^{n_x \times n_d}$ and $C_d \in \mathbb{R}^{n_y \times n_d}$ are chosen appropriately such that the following condition holds true for detectability.

$$rank \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n_x + n_y \tag{8.24}$$

Here n_x, n_y, n_d are the number of states, outputs and disturbance variables of the system.

The linear MPC (LQ optimal control structure + receeding horizon strategy) should be constructed with the augmented model of Equation 8.23 and 8.24. If the conditions for detectability holds true, then the resulting model predictive controller should produce offset free outputs i.e. integral action.

ii) [5%] Now if we assume that the disturbances acting on the system are unknown and not directly measurable, can we still design an MPC with integral action? If yes, describe how this can be done.

Suggested solution:

Yes we can still design an MPC with integral action even if the disturbances acting on the system are not measured or are unknown. From equation 8.23 and 8.24 it is clear that if one or some of the elements of the extended states $\tilde{x}_k$ are not measurable, then they should be estimated assuming that they are observable. Here a standard Kalman filter algorithm for linear system can be used to estimate the augmented states $\tilde{x}_k = \begin{bmatrix} x_k \\ d_k \end{bmatrix}$ which also includes the disturbances as the added states to the system. The estimated extended states can then be fed to the MPC.

Problem 3

Tasks:

i) (15%) Explain it in details how Lagrange multipliers can be used to reduce the size of the optimal control of equation (3).

Suggested solution:

The optimal control problem taken into consideration is,

$$\min_{z} \quad \frac{1}{2}z^T Hz + c^T z, \qquad (70)$$

$$s.t$$

$$A_\varepsilon z = b_\varepsilon,$$

where z is a vector that contains the unknowns. The Lagrangian function $F(z, \lambda)$ is defined as,

$$\mathrm{F(z, \lambda) = f(z) + \lambda^T(A_\epsilon z - b_\epsilon)} \qquad (71)$$

where λ is known as Lagrange multiplier. The reduced optimal control problem (without equality constraints) is then,

$$\min_{(z,\lambda)} \quad \mathrm{F(z,\lambda)} = \frac{1}{2}z^T Hz + c^T z + \lambda^T(A_\epsilon z - b_\epsilon) \qquad (72)$$

To find the minimum of unconstrained problem of equation (72), we simply take the first derivative and equate them to zero. Also note that equation (72) has two unknown variables to be optimized which are z and λ so we need to take partial derivative as,

$$\frac{\partial F(z,\lambda)}{\partial z} = Hz + c + A_\epsilon^T\lambda \qquad (73)$$

$$\frac{\partial F(z,\lambda)}{\partial \lambda} = A_\epsilon z - b_\epsilon \qquad (74)$$

Equating (73) and (74) to zero for minimum we get,

$$Hz + A_\epsilon^T\lambda = -c \qquad (75)$$

$$A_\epsilon z + 0\lambda = b_\epsilon, \qquad (76)$$

Arranging (75) and (76) in compact form we get,

$$\underbrace{\begin{bmatrix} H & A_\epsilon^T \\ A_\epsilon & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} z \\ \lambda \end{bmatrix}}_{\tilde{z}} = \underbrace{\begin{bmatrix} -c \\ b_\epsilon \end{bmatrix}}_{\tilde{b}_\epsilon} \qquad (77)$$

Equation (77) is a linear algebraic equation and can be solved to find the optimal solution of the original problem of (70). Assuming that M is invertible, optimal solution $\tilde{z}^* = M^{-1}\tilde{b}_\epsilon$. In another form (which is also an accepted answer) is, $\tilde{z}^* = (M^T M)^{-1}M^T\tilde{b}_\epsilon$.

ii) (5%) What happens to the size of the unknowns to be optimized after the reduction? Comment about it.

Suggested solution:

The size of the unknowns to be optimized is increased. Originally, the variables to be optimized were only z. Using Lagrangian functions, the variables to be optimized are z and λ (in addition).