Suggested Solutions 2024

Problem 1

MPC can also be used as an optimal controller to control a process where multiple objectives have to be taken into consideration simultaneously. Let us assume that you are working with a process where there are three objectives functions that are conflicting to each other.
Let us denote the three objectives as $F_1(x), F_2(x)$ and $F_3(x)$. Here x are the decision variables. Assume that objective function $F_2(x)$ is the most important objective function, $F_3(x)$ is the second most important objective, and $F_1(x)$ is the least important objective function among these three conflicting objective functions.

Tasks:

i) [15%] How would you used the hierarchical method to solve this multi objective control problem? Explain and show mathematical formulation.

Suggested Solution:

With the lexicographic or hierarchical method, the objective functions are arranged in the order of importance with the highest prioritized objective at the top. From the problem description this will be $[F_2(x), F_3(x), F_1(x)]$. One after the other, each objective function is minimized starting with the most important one and proceeding forward according to the order of importance. At any given step, the prioritized objectives form inequality constraints in order not to sacrifice its performance.

Step1:

Since $F_2(x)$ is the most important objective, in the first step, this objective function is minimized as,

$$\min_{x} \quad F_2(x) \tag{1}$$

$$s.t$$

$$< process\ constraints > \tag{2}$$

Here <process constraints> denote any equality or inequality constraints associated with the problem. Now after solving this optimization problem of step (1) we obtain optimal values x^* and corresponding optimal functional value $F_2(x^*)$

Step2:

Now since $F_3(x)$ is the second most important objective, in the second step, this objective function is minimized. However, an extra constraint (equation (5)) must be added so that all the prioritized objective function appearing above it (in this case $F_2(x)$) do not have to sacrifice it's optimal performance.

$$\min_{x} \quad F_3(x) \tag{3}$$

$$s.t$$

$$< process\ constraints > \tag{4}$$

$$F_{2(x)} \leq F_2(x^*) \tag{5}$$

After solving this optimization problem given by equations (3), (4) and (5), we will get a new optimal values of the decision variable x^* and corresponding optimal functional value $F_3(x^*)$.

Step3:

Finally since $F_1(x)$ is the least important objective, in the last step, this objective function is minimized. However, two extra constraint (equation (5), and equation (6)) must be added so that all the prioritized objective function appearing above it (in this case $F_2(x)$ and $F_3(x)$) do not have to sacrifice their optimal performance.

$$\min_{x} \quad F_3(x) \tag{6}$$

$s.t$

$$< process\ constraints > \tag{7}$$
$$F_{2(x)} \leq F_2(x^*) \tag{8}$$
$$F_{3(x)} \leq F_3(x^*) \tag{9}$$

Finally after solving the last optimization problem given by equations (6), (7), (8) and (9), we will once again get a new optimal value of the decision variable x^*. This will be the final optimal value of the whole multi-objective optimization problem. To make an MPC, all these three steps should be re-evaluation at each sampling time and the standard receeding horizon strategy has to be used.

ii) [5%] Explain briefly a potential drawback or disadvantage of this method.

Suggested Solution:

A potential drawback of this method is that normally objectives with lower priorities will not be properly satisfied. Further, we obtain one optimum for a given lexicographic order.

Problem 2

With the Kronecker product formulation of an LQ optimal control problem, the size of the control problem can easily become very large. The size of the control problem can be reduced by elimination of the equality constraints.

Let us consider an LQ optimal control problem expressed in the standard QP form as,

$$\left.\begin{aligned} &\underset{z}{Min} \ \frac{1}{2} z^T H z + c^T z \\ &\text{subject to} \\ &\qquad A_e z = b_e \\ &\qquad A_i z \leq b_i \\ &\qquad z_L \leq z \leq z_H \end{aligned}\right\} \tag{1}$$

The problem given by equation (1) can be converted into a QP problem of the reduced form as,

$$\left.\begin{aligned} &\underset{z_2}{min} \ \frac{1}{2} z_2^T \tilde{H} z_2 + \tilde{c}^T z_2 \\ &\qquad \tilde{A}_i z_2 \leq \tilde{b}_i \end{aligned}\right\} \tag{2}$$

Tasks

(i) [15%] Explain and show in detail how QR factorization can be used to eliminate the equality constraints in order to reduce the size of the optimal control problem. Include all necessary calculations and equations.

Suggested solution:

The idea here is to use the linear equality constraint $A_e z = b_e$ present in the original problem to split the unknown variables z into two parts: "basic variables" and "non-basic variables". Using the equality constraint, the basic variables are expressed in terms of the non-basic variables. The basic variables (which are the functions of non-basic variables) are then substituted in the objective function of equation. The objective function will then only have the non-basic variables. This will also result in the elimination of the linear equality constraint & we will obtain the reduced problem.

Let us consider the linear equality constraint as,

$$A_e z = b_e \tag{15}$$

Let us assume equation (15) has n number of equations i.e. it is a compact form of n linear equality constraints. Then, $A_e \epsilon R^{n \times n_z}$, where n_z is the total number of unknown variables that are listed in vector z. Let the rank of A_e be r i.e. $r = \text{rank}(A_e)$. Rank of a matrix gives you the number of linearly independent rows of the matrix, here in this case, the number of independent linear equality constraints.

Now, let us decompose A_e into the product of Q and R as,

$$A_e = QR$$

where Q = orthogonal matrix i.e. $Q^T Q = I$ and $Q \epsilon \mathbb{R}^{n \times n}$, R = upper triangular matrix, $R \epsilon \mathbb{R}^{n \times n_z}$.

Furthermore, R can be written as,

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \tag{16}$$

where, $R_1 \epsilon \mathbb{R}^{r \times r}$ is full rank i.e. $\text{rank}\,(R_1) = r = \text{rank}(A_e)$ and $R_2 \epsilon \mathbb{R}^{r \times (n_z - r)}$ is the upper right submatrix of R.

Note that the 0's under R_1 has the size of $(n - r) \times r$ and the 0's under R_2 has the size of $(n - r) \times (n_z - r)$. So, we can write the linear equality constraints of equation (15) as,

$$A_e z = b_e$$

$$QRz = b_e$$

Multiplying both sides by Q^T we get,

$$\underbrace{Q^T Q}_{I}\, Rz = \underbrace{Q^T\, b_e}_{\bar{b}_\epsilon}$$

$$Rz = \bar{b}_e \tag{17}$$

Now let us split the vector of unknowns z into $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ or $z^T = (z_1^T, z_2^T)$

where, z_1 are the basic variables of size $(r \times 1)$ and z_2 are the nonbasic variables of size $(n_z - r) \times 1$.

Also let us split $\overline{\mathrm{b}}_{\mathrm{e}}$ into two parts as,

$$\overline{b}_e = \begin{bmatrix} \overline{b}_{e,1} \\ \overline{b}_{e,2} \end{bmatrix} \tag{18}$$

where, $\overline{b}_{e,1} \epsilon \mathbb{R}^{r\times 1}$ and $\overline{b}_{e,2} \epsilon \mathbb{R}^{(n-r)\times 1}$

Then we have from equations (4.24, 4.25 & 4.26) we get,

$$\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \overline{b}_{e,1} \\ \overline{b}_{e,2} \end{bmatrix} \tag{19}$$

From equation 19 we get,

$$R_1 z_1 + R_2 z_2 = \overline{b}_{e,1}$$

Since R_1 is full ranked, it is invertible. Then we can express z_1 (the basic variables) in terms of z_2 (the non-basic variables) as,

$$z_1 = R_1^{-1} \left(\overline{b}_{e,1} - R_2 z_2\right) \tag{20}$$

So, in summary we have,

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} R_1^{-1} \left(\overline{b}_{e,1} - R_2 z_2\right) \\ z_2 \end{bmatrix} = \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} - R_1^{-1} R_2 z_2 \\ 0 + z_2 \end{bmatrix}$$

$$\downarrow$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \tag{21}$$

The sizes of zeros and identity matrix in equation (21) are: $0 \rightarrow (n_z - r) \times 1$ and $I \rightarrow (n_z - r) \times (n_z - r)$.

Now let us substitute z from equation (21) in the original problem given by equation (4.13) i.e.

$\Rightarrow \frac{1}{2} z^T H z + c^T z$ can be written as,

$$\frac{1}{2} \left(\begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \right)^T H \left(\begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \right) + c^T \left(\begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \right) \tag{22}$$

We can see that equation (22) has only the non-basic variable z_2.

Comment: Many students do not show the intermediate calculations for $\widetilde{H}, \tilde{c}$ and $\widetilde{K}$ as shown below from equation (22a) to (22e).

To solve equation (22) we can make use of the matrix transpose properties. If for example A and B are two matrices then, matrix transpose properties say that $(A+B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$.

Then we have from equation (22),

$$\frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T + z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T\right) H \left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2\right) + c^T \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + c^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 \qquad (22a)$$

We can further solve equation 22(a) as,

$$\frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 + z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2\right) + c^T \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + c^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 \qquad (22b)$$

Then we can further go on solving 22(b) as,

$$\frac{1}{2}\left(z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2\right) + \frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}\right) + c^T \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 + z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}\right) + c^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 \qquad (22c)$$

We know that H is a symmetric diagonal matrix, i.e. $H^T = H$. Then if A and B are any two matrices, then from matrix transpose property, we get, $A^T H\, B = B^T H\, A$. Thus the second term of the "green coloured" expression $z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}$ can also be written as $\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2$. Then we have from equation (22c),

$$\frac{1}{2}\left(z_2^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2\right) + \frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}\right) + c^T \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 + c^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} z_2 \qquad (22d)$$

From the "green coloured" part of equation 22(d) we have z_2 as common. Then we have,

$$\frac{1}{2}\left(z_2^T \underbrace{\begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}}_{\tilde{H}} z_2\right) + \underbrace{\frac{1}{2}\left(\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}\right) + c^T \begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}}_{\tilde{K}} + \left(\underbrace{\begin{bmatrix} R_1^{-1}\,\bar{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix} + c^T \begin{bmatrix} -R_1^{-1}\,R_2 \\ I \end{bmatrix}}_{\tilde{c}^T}\right) z_2 \qquad (22e)$$

In compact form, equation (22e) can be expressed as,

$$\min_{z_2},\ \frac{1}{2}\, z_2^T\, \tilde{H} z_2 + \tilde{c}^T z_2 + \tilde{K} \qquad (23)$$

where,

$$\widetilde{H} = \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} R_1^{-1} R_2 \\ I \end{bmatrix} \tag{24}$$

$$\tilde{c}^T = \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} + c^T \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix}$$

Or by using transpose property of matrices $(A+B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$ (25)

$$\tilde{c} = \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix}^T c$$

$$\widetilde{K} = \frac{1}{2} \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix}^T H \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} + c^T R_1^{-1} \begin{bmatrix} \overline{b}_{e,1} \\ 0 \end{bmatrix} = \text{constant} \tag{26}$$

Since $\widetilde{K}$ is constant, it can be safely removed from the optimization problem. Then the objective of the reduced problem is,

$$\min_{z_2} \; \frac{1}{2} z_2^T \widetilde{H} z_2 + \tilde{c}^T z_2$$

At this stage we have eliminated the equality constraint. But the original optimization problem also has linear inequality constraints and bounds.

For the inequality constraints we have,

$$A_i z \leq b_i$$

Substituting z from equation (21) in this above equation we get,

$$A_i \left(\begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \right) \leq b_i$$

$$A_i \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \leq b_i - A_i \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} \tag{27}$$

For the bounds, we have ,

$$z_L \leq z \leq z_H$$

Substituting z from equation (21) in this above equation we get,

$$z_L \leq \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} + \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \leq z_H$$

$$z_L - \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} \leq \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \leq z_H - \begin{bmatrix} R_1^{-1}\, \overline{b}_{e,1} \\ 0 \end{bmatrix} \tag{28}$$

Equation (28) can be written separately using two equations (i.e. bounds can be expressed as two inequality constraints) as,

$$-\begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \le -z_L + \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} z_2 \le z_H - \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} \tag{30}$$

Equations (28), (29) & (30) can be written in compact form as,

$$\tilde{A}_i z_2 \le \tilde{b}_i$$

where,

$$\tilde{A}_i = \begin{bmatrix} A_i \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} \\ -\begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} \\ \begin{bmatrix} -R_1^{-1} R_2 \\ I \end{bmatrix} \end{bmatrix}, \quad \tilde{b}_i = \begin{bmatrix} b_i - A_i \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} \\ -z_L + \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} \\ z_H - \begin{bmatrix} R_1^{-1} \overline{b}_{e,1} \\ 0 \end{bmatrix} \end{bmatrix} \tag{31}$$

Thus, the original problem has been formulated as the reduced problem as,

$$\min_{z_2} \ \frac{1}{2} z_2^T \widetilde{H} z_2 + \tilde{c}^T z_2 \tag{32}$$

subject to,

$$\tilde{A}_i z_2 \le \tilde{b}_i \tag{33}$$

where $\widetilde{H}$ is given by equation (24), $\tilde{c}$ is given by equation (25), $\tilde{A}_i$ and $\tilde{b}_i$ are given by equation (31).

The reduced problem given by equations (32 & 33) does not have equality constraints but it is equivalent to the original problem. We can solve the reduced problem of equations (32 & 33) using qpOASES solver in Simulink or quadprog solver in MATLAB.

Let us assume that the optimal values returned by the solver be denoted by z_2^* . After finding the optimal values z_2^* of the reduced optimization problem of (32 & 33), we can find the optimal values of the basic variables as,

$$z_1^* = R_1^{-1}\left(\overline{b}_{\epsilon,1} - R_2 z_2^* \right) \tag{34}$$

Finally, we have the optimal values of the unknown variables as, $z^* = \begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix}$

(ii) [5%] What are the advantages and the disadvantages of reducing the size of an optimal control problem.

Suggested Solution:

When the size of an optimal control problem is reduced the control problem becomes compact and is usually solved faster by an optimization solver which does not support sparsity. For embedded systems, such a compact optimal control problem is advantageous. This is because of the lack of higher computational power on embedded systems. So the compact is the LQ optimal control problem, the faster it can be solved in an embedded system. However, reduced control problem size loses sparsity, and solvers that support sparsity will not be able to take the benefit of it.

Problem 3

The design of a standard MPC formulation cannot guarantee feasibility, i.e. it cannot be guaranteed that a feasible solution always exists. However, problems caused by infeasibility can be improved. A mathematical formulation of a linear MPC problem with output constraints is shown below,

$$\min_{z} \quad \frac{1}{2} z^T H z + c^T z, \tag{3}$$

$s.t$

$$A_\varepsilon z = b_\varepsilon, \tag{4}$$

$$y_L \leq y \leq y_H \tag{5}$$

where z is a vector that contains the unknowns or decision variables. Equation (3) is the objective function, equation (4) is the linear equality constraints and equation (5) is the output constraints or bounds on the outputs.

Tasks:

i) (15%) Explain in details how you can make use of slack variables to improve the feasibility of the control problem. Show also the mathematical formulation of the relaxed problem.

Suggested solution:

The presence of output constraints in an optimization problem may lead to the problem being infeasible under certain operating conditions. When problem becomes infeasible, the optimizer will not be able to find out any solution to the optimization problem and hence may either crash or produce garbage results. A simple way to handle infeasibility due to the presence of output constraints is to make use of the slack variables to formulate a more relaxed problem. For this the lower and the upper bounds of the output variables can be dynamically changed or adjusted to bring the problem back to feasibility. Thus equation (5) from the problem description which is the output constraint can be modified as,

$$y_L - S_L \leq y \leq y_H + S_H \tag{3a}$$

Here the variables S_L and S_H are called the slack variables. The values for these slack variables cannot be chosen arbitrarily. Relaxation of the constraints should be avoided if possible, but if necessary then the constraints should be relaxed as gently as possible so that the constraint violation which often is temporary is as lower as possible. Thus the slack variables have to be adjusted dynamically. For this, these slack variables can be considered to be extra decision variables for the optimization problem in addition to the already existing decision variable z.

In addition, the use of the slack variables should be penalized so that they are only used when it is utmost necessary, i.e. when the optimization problem tends to run into infeasibility. Under normal condition, the use of slack variables should be avoided. For this to happen, the objective function should contain the penalization of the slack variables as an additional term.

Thus, to improve the feasibility, the original problem can be reformulated as shown below.

$$\min_{(z, S_L, S_H)} \quad \frac{1}{2} z^T H z + c^T z + \beta_L S_L + \beta_H S_H, \tag{3b}$$

$s.t$

$$A_\varepsilon z = b_\varepsilon, \tag{3c}$$
$$y_L - S_L \leq y \leq y_H + S_H \tag{3d}$$

As can be seen from the reformulated problem, the slack variables are added to the list of decision variables. The objective function is modified to introduce penalty on the slack variables and the output constraints is relaxed using the slack variables. β_L and β_H are the weighting matrices for the slack variables. The optimizer will appropriately find a suitable value for S_L and S_H such that the constraints are violated in the most gentle manner. Usually, the values of S_L and S_H will be zero if no output constraints are being violated and the problem is feasible. The slack variables should be non-zero if and only if the problem tends to run into infeasibility.

ii) (5%) Explain in your own words concept about hard and soft constraints. Give some examples of each.

Suggested Solution:
(i) Hard constraint:
Constraints that have to be always obeyed strictly are the hard constraints. A system must adhere to hard constraints. Usually constraints on the input variables can be posed as hard constraints. For example: the opening of a choke valve in a pipeline should be within 0 & 100% i.e. $0 \leq u \leq 100$ is a hard constraint. This constraint cannot be violated at any cost. The valve cannot be opened more than 100% & cannot be closed below 0% (the physical structure & the operational condition of the choke valve strictly puts this limit). Any value of u outside $0 \leq u \leq 100$ is simply not possible/feasible. Other examples of hard constraints can be: capacity of an equipment, limits on actuators, etc.

(ii) Soft constraints:
Constraints which are fulfilled if possible, but if it is not possible, disobeying or breaking the constraints is also allowed are the soft constraints. However, violating the constraint should be made as gentle as possible. With soft constraints, the system tries to adhere or stick to it, but the system can violate the constraints if necessary in order to find a feasible solution (but of course a solution that complies with the hard constraints).
For example: Process outputs like flow rate, temperature, pressure etc. (unless they are too serious to disobey or too serious to violate) can be regarding as soft constraints depending on the operating conditions. Also note that when disobeying or breaking the constraints, you may have to compromise some other things like quality of the product. Violating the soft constraint is also known as relaxing the constraints. The goal during constraint relaxation is to minimize the total amount of violation of all the soft constraints

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Roshan Sharma
12th November 2024, Porsgrunn, Norway